Topological cyclic homology of Elliptic cohomology

ECHT Seminar

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joint with

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[A-KACHR]

Notation:

$BP\langle n\rangle$ $\quad BP\langle n\rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ $\quad V(n) = \mathbb{S}/(p, \dots, v_n)$

$(n \geq -1$, p prime $\quad p = v_0$

n	$BP\langle n\rangle$	
-1	$H\mathbb{F}_p$	E_∞-ring
0	$H\mathbb{Z}_{(p)}$	E_∞-ring
1	ℓ summand of $ku_{(p)}$	E_∞-ring
2	Summand of $tmf_{(p)}$	$p \geq 5$

n	Exists	Ring
0	$p \geq 2$	$p \geq 3$
1	$p \geq 3$	$p \geq 5$
2	$p \geq 5$	$p \geq 7$
3	$p \geq 7$	$p \geq 11$

$v_4 \in V(3)_{2p^4-2}$ not known

E_3-BP-alg by [Thm A Hahn-Wilson]

Theorem [A-K ACHR]

There is an isomorphism of $P(v_3)$-modules

$$
\begin{aligned}
V(2)_* TC(BP\langle 2\rangle) \cong\ & P(v_3) \otimes E(\lambda_1, \lambda_2, \lambda_3, \partial) \\
& \oplus P(v_3) \otimes E(\lambda_2, \lambda_3) \otimes \mathbb{F}_p\{\Xi_{1,d} : 0<d<p\} \\
& \oplus P(v_3) \otimes E(\lambda_1, \lambda_3) \otimes \mathbb{F}_p\{\Xi_{2,d} : 0<d<p\} \\
& \oplus P(v_3) \otimes E(\lambda_1, \lambda_2) \otimes \mathbb{F}_p\{\Xi_{3,d} : 0<d<p\}
\end{aligned}
$$

$|v_3| = 2p^3 - 2$, $|\lambda_i| = 2p^i - 1$, $|\partial| = -1$, and

$|\Xi_{i,d}| = 2p^i - 2p^{i-1}d - 1$ (corrected typo.)

Cor. [A-K ACHR]

There is an exact sequence

$$0 \to \Sigma^{-2}\mathbb{F}_p\{\bar{\tau}_1,\bar{\tau}_2,\bar{\tau}_3\} \to V(2)_* K(BP\langle 2\rangle^{\wedge}_p) \longrightarrow V(2)_* TC(BP\langle 2\rangle) \to \Sigma^{-1}\mathbb{F}_p \to 0$$

Cor. [A-K ACHR]

The canonical map

$$V(2)_s K(BP\langle 2\rangle) \longrightarrow v_3^{-1} V(2)_s K(BP\langle 2\rangle)$$

is an isomorphism for $s > 2p^2+2p-1$.

Motivation (Height 0 to Height 1) ③

Suslin rigidity [Suslin 1984]

$$V(0) \wedge K(\overline{\mathbb{Q}}_p) \simeq V(0) \wedge ku$$

$$\left(V(0)_* ku = P(b) \quad b^{p-1} = v_1 \right)$$

[Bökstedt – Madsen 1995]

$$V(0)_* TC(\mathbb{Z}_{(p)}) \cong P(v_1) \otimes E(\lambda_1, \partial)$$
$$\oplus P(v_1) \otimes \mathbb{F}_p\{ \Xi_{1,d} : 0 < d < p \}$$

used to verify a case of

Lichtenbaum – Quillen conjecture

$$G = Gal(\overline{\mathbb{Q}}_p / \mathbb{Q}_p$$

$$\begin{array}{ccc} H^{-s}_{Gal}(\mathbb{Q}_p ; \mathbb{F}_p(t/2)) & \Longrightarrow & V(0)_{s+t} K(\overline{\mathbb{Q}}_p)^{hG} \\ \uparrow & & \uparrow \\ H^{-s}_{mot}(\mathbb{Q}_p ; \mathbb{F}_p(t/2)) & \longrightarrow & V(0)_{s+t} K(\mathbb{Q}_p) \end{array}$$

0 – connected

(Height 1 to Height 2) ④

[Ausoni–Rognes 2002]

$$V(1)_* K(BP\langle 1\rangle) \cong P(v_2) \otimes E(\lambda_1, \lambda_2, \partial)$$
$$\oplus\, P(v_2) \otimes E(\lambda_2) \otimes \mathbb{F}_p\{\Xi_{1,d} : 0<d<p\}$$
$$\oplus\, P(v_2) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{\Xi_{2,d} : 0<d<p\}$$

[Rognes 2014] suggests that there is an analogous story

$$\begin{array}{ccc} H^{-s}_{Gal}(\mathrm{ff}L_p \,;\, \mathbb{F}_{p^2}(t/2)) & \Longrightarrow & V(1)_{s+t} K(\mathcal{L}_1)^{hG} \\ \uparrow & & \uparrow \\ H^{-s}_{mot}(\mathrm{ff}L_p \,;\, \mathbb{F}_{p^2}(t/2)) & \Longrightarrow & V(1)_{s+t} K(\mathrm{ff}L_p) \end{array}$$

coconnected?

$L_p = BP\langle 1\rangle^{\wedge}_p[v_1]$ $G = \mathrm{Gal}(\mathcal{L}_1/\mathrm{ff}L_p)$

$\mathrm{ff}L_p \to \mathcal{L}_1$ some maximal extension

based on computations with Ausoni.

Red-shift

Def. We say X (bdd below, p-complete) has pure fp-type n if $F(n-1)\otimes X$ is a finitely generated free $P(v_n^d)$-module for some finite, type n spectrum $F(n-1)$.

Pure fp-type red-shift Conj [Rognes 2000]

If R is a (regular) $\mathbb{E}_1$-ring with pure fp-type n, then $TC(R)$ has pure fp-type $n+1$.

Ex.

Height	R	Authors
0 to 1	$H\mathbb{Z}_{(p)}$	Bökstedt-Madsen 1995
1 to 2	...., ℓ, ku, $k(1)$	Ausoni-Rognes 2002, Ausoni 2011, Ausoni-Rognes 2012
2 to 3	$BP\langle 2\rangle$	A-K CHR

<u>Def</u>. We say X has **telescopic complexity n** if the canonical map
$$F(n-1)_s X \longrightarrow v_n^{-1} F(n-1)_s X$$
is an isomorphism for $s >> 0$.

<u>Rem</u>. The pure fp-type red-shift conj implies the following weaker form of the red-shift conjecture.

<u>Telescopic complexity red-shift conj.</u> [Ausoni–Rognes 2008]

If R is a (regular) $\mathbb{E}_1$-ring with telescopic complexity n then $K(R)$ has telescopic complexity $n+1$.

<u>Rem</u>. [Thm B. Hahn–Wilson 2020] proved the telescopic complexity red-shift conj. for $BP\langle n\rangle$ $\forall p, n$.

Outline of proof

Step 1: Compute

$$V(2)_*THH(BP\langle 2\rangle) = E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu)$$

$$|\lambda_i| = 2p^i-1,\ |\mu| = 2p^3.$$

Step 2: (Segal conjecture)

Prove that the Tate valued Frobenius map

$$☆\quad V(2)_*THH(BP\langle 2\rangle) \xrightarrow{\varphi_p} V(2)_*THH(BP\langle 2\rangle)^{tC_p}$$

can be identified with the localization map

$$E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu) \longrightarrow E(\lambda_1,\lambda_2,\lambda_3)\otimes P(\mu^{\pm 1}).$$

In particular, ☆ is an isomorphism

for $* > 2p^2+2p-3$.

Step 3: Bootstrap up to TC^- and TP

$V = V(2)$, $T = THH$

$$\begin{tikzcd}
V \wedge T(BP\langle 2\rangle)^{C_{p^n}} \arrow[r] \arrow[d, "\Gamma_n"] & V \wedge T(BP\langle 2\rangle)^{C_{p^{n-1}}} \arrow[d, "\hat{\Gamma}_n"] \\
V \wedge T(BP\langle 2\rangle)^{hC_{p^n}} \arrow[r] & V \wedge T(BP\langle 2\rangle)^{tC_{p^n}}
\end{tikzcd}$$

$(2p^2+2p-3)$-coconnected

by Step 2 and Tsalidis' Theorem.

$$\begin{tikzcd}
V_* TC^-(B) & V_* TF(B) \arrow[l] \arrow[r] & V_* TP(B) \\
V_* T(B)^{hC_{p^n}} \arrow[u, dotted, leftarrow] \arrow[d, "F^h"] & V_* T(B)^{C_{p^n}} \arrow[l, "\Gamma_n"'] \arrow[r, "\hat{\Gamma}_n"] \arrow[u, dotted, leftarrow] \arrow[d, "F"] & V_* T(B)^{tC_{p^{n+1}}} \arrow[u, dotted, leftarrow] \arrow[d, "F^t"] \\
V_* T(B)^{hC_{p^{n-1}}} \arrow[d, dotted, "F^h"] & V_* T(B)^{C_{p^{n-1}}} \arrow[l] \arrow[r] \arrow[d, dotted, "F"] & V_* T(B)^{tC_{p^n}} \arrow[d, dotted, "F^t"] \\
V_* T(B)^{hC_p} \arrow[d, "F^h"] & V_* T(B)^{C_p} \arrow[l, "\Gamma_1"'] \arrow[r, "\hat{\Gamma}_1"] \arrow[d, "F"] & V_* T(B)^{tC_{p^2}} \arrow[d, "F^t"] \\
V_* T(B) & V_* T(B) \arrow[l, "id"'] \arrow[r] & V_* T(B)^{tC_p}
\end{tikzcd}$$

Step 4 : (Homotopy equalizer

cyclotomic structure

$$TC(BP\langle 2\rangle) := \mathrm{hoeq}\left(TF(BP\langle 2\rangle) \underset{1}{\overset{R}{\rightrightarrows}} TF(BP\langle 2\rangle)\right)$$

$$TF(R) = \lim_F \left(\cdots \to T(R)^{C_p} \xrightarrow{F} T(R)\right)$$

inclusion of fixed points

[Nikolaus-Scholze 2018]

$$TC(BP\langle 2\rangle) \simeq \mathrm{hoeq}\left(TC^-(BP\langle 2\rangle) \underset{G^{-1}\circ\varphi}{\overset{can}{\rightrightarrows}} TP(BP\langle 2\rangle)\right)$$

$$\begin{tikzcd}
 & V_* TC^-(BP\langle 2\rangle)[t^{-1}] \arrow[d, equal] \\
 & V_* TP(BP\langle 2\rangle) \arrow[d, "G_*", "\cong"'] \\
V_* TC^-(BP\langle 2\rangle) \arrow[ur, bend left, "can_*"] \arrow[r, bend right, "\varphi_*"'] & V_* \left(T(BP\langle 2\rangle)^{tC_p}\right)^{hT} \arrow[d, equal, "Thm"] \\
 & V_* TC^-(BP\langle 2\rangle)[\mu^{-1}]
\end{tikzcd}$$

interestingly isomorphism

<u>Rem</u>. Before Nikolaus-Scholze, TC was computed the same way, but we only knew it gave the correct answer above the bound given by the Segal conjecture.

<u>Ex</u>.

$$\begin{tikzcd}
V(2)_* TF(BP\langle 2\rangle) \arrow[r, "R"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & V(2)_* TF(BP\langle 2\rangle) \arrow[d] \\
V(2)_* TC^-(BP\langle 2\rangle) \arrow[r, "can"] & V(2)_* TP(BP\langle 2\rangle)
\end{tikzcd}$$

$(2p^2+2p-3)$-coconnected

We also know the map

$$V_* TC(BP\langle 2\rangle) \to V_* TC(BP\langle 1\rangle)$$

is $(2p^2-1)$-connected

This leaves a gap in degrees $[2p^2-1, 2p^2+2p-3]$.

For $BP\langle 0\rangle$ and $BP\langle 1\rangle$ there is no gap.

Step 4 cont.

$* \geq 0$

$t_m = v_3$

$$V_* TC^-(BP\langle 2\rangle) = A \oplus B \oplus C \qquad * \geq 0$$

$$A = E(\lambda_1, \lambda_2, \lambda_3) \otimes P(t_m) \qquad C = \text{Doesn't contribute to } TC$$

$$B = \prod_{n \geq 0} v_3\text{-torsion-modules} \qquad B' = \prod_{n \geq 0} v_3\text{-torsion modules}$$

$$V_* TP(B) = A \oplus B' \oplus C$$

$$\begin{tikzcd}
A \arrow[r] & A \arrow[r, shift left, "id"] \arrow[r, shift right, "id"'] & A \arrow[r] & A \\
\oplus & & & \\
eq(B \rightrightarrows B') \arrow[r] & B \arrow[r, shift left, "T"] \arrow[r, shift right, "id"'] & B' \arrow[r] & 0 \\
\oplus & \oplus & & \\
0 \arrow[r] & C \arrow[r, shift left, "0"] \arrow[r, shift right, "id"'] & C \arrow[r] & 0
\end{tikzcd}$$

$$V_* TC(BP\langle 2\rangle) = \tilde{A} \oplus \tilde{A}\{d\} \oplus eq(B \rightrightarrows B')$$

Proof sketch of Step 2.

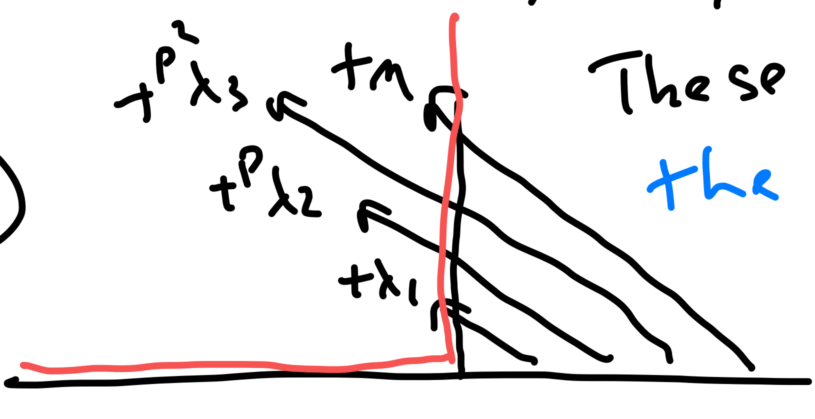
$$V(2)_*\mathbb{S} \ni \alpha_1,\ \beta_1',\ \gamma_1''\ v_3$$

$$\begin{tikzcd}
V(2)_*\mathbb{S} \arrow[d] & & \\
V(2)_*K(BP) \arrow[d] & & \\
V(2)_*K(BP\langle 2\rangle) \arrow[d] & & \\
V(2)_*TC(BP\langle 2\rangle) \arrow[d] & & \\
V(2)_*T(BP\langle 2\rangle)^{C_p} \arrow[r, "R"] \arrow[d, "\Gamma_1"] & V(2)_*T(BP\langle 2\rangle) \arrow[d, "\hat{\Gamma}_1 = \varphi_p"] \\
V(2)_*T(BP\langle 2\rangle)^{hC_p} \arrow[r, "R^h = can"'] & V(2)_*T(BP\langle 2\rangle)^{tC_p}
\end{tikzcd}$$

$$\Downarrow$$

$$t\lambda_1,\ t^p\lambda_2,\ t^{p^2}\lambda_3,\ t\mu$$

$\Longrightarrow$



These generate all the differentials.

$\hat{H}^*(C_p; V_*T(BP\langle 2\rangle))$

0

To prove that

$$\begin{aligned} \alpha_{1,1} & \\ \beta_1' & \longmapsto \\ \gamma''_{1,1} & \\ v_3 & \end{aligned} \qquad \begin{aligned} &+\lambda_1 \\ &+^p\lambda_2 \\ &+^{p^2}\lambda_3 \\ &+m \end{aligned}$$

we construct

homotopy Power operations

$$\mathcal{P}_0^k : \pi_{2k-1} R \to V(0)_{2pk-1} R$$

and

$$\mathcal{P}_1^k : V(0)_{2k-1} R \to V(1)_{2pk-1} R$$

for $\mathbb{E}_2$-rings R

Ex.

$$R \in \{T(BP\langle 2\rangle), TC(BP\langle 2\rangle), K(BP\langle 2\rangle)\}$$

We also prove a Cartan formula (in the presence of extra structure) so that

$$P_0^{p}(t\lambda_1) = t^p P_0^{p}(\lambda_1) = t^p \lambda_2$$

and

$$P_1^{p^2}(t\lambda_2) = t^{p^2} P_1^{p^2}(\lambda_2) = t^{p^2}\lambda_3$$

in $V(2)_* TC^-(BP)$ and

$$P_0^p(\alpha_1) = \beta_1' \ , \quad P_1^{p^2}(\beta_1') = \gamma_1''$$

in $V(2)_*$.

Finally, we use the m-th-order approximate homotopy fixed point spectral sequences

$$\mathbb{Z}[t]/t^{m+1} \otimes V_* T(BP) \Rightarrow V_* F(E\Pi_+^{(m)}, T(BP))^{\Pi}$$

$$\mathbb{Z}[t]/t^{m+1} \otimes V_* T(BP\langle 2\rangle) \Rightarrow V_* F(\underbrace{E\Pi_+^{(m)}, T(BP\langle 2\rangle)}_{Y})$$

$$\mathbb{Z}[t]/t^{m+1} \otimes H_* V \wedge T(BP\langle 2\rangle) \Rightarrow H_* V \wedge Y$$

to conclude that

$$\alpha_1 \overset{\in V_*}{\longmapsto} t\lambda_1 \in V_* T\bar{C}(BP\langle 2\rangle).$$

$$\beta_1' \longmapsto t^p \lambda_2$$

$$\gamma_1' \longmapsto t^{p^2} \lambda_3$$

$$v_3 \longmapsto t\mu$$

$\square$

Thanks!