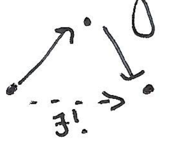
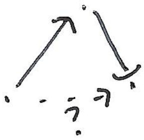
Talk 1: Introduction to 1-Segal and 2-Segal sets and spaces

1-Segal sets: behave like categories: have objects $\bullet$, morphisms $\bullet \to \bullet$, unique composition , which is associative.

2-Segal sets: weaker structure: composition may not exist or be unique  , but still associative.

Goal today: Introduce and define 1-Segal and 2-Segal sets/spaces and give several examples.

To begin: crash course in simplicial sets

category Δ: objects finite ordered sets $[n] = \{0 \leq 1 \leq \ldots \leq n\}$ for $n \geq 0$

morphisms order-preserving

$$[0] \rightrightarrows [1] \quad [2] \quad \cdots$$

opposite category Δ^{op}: $[0] \leftleftarrows [1] \quad [2] \quad \cdots$

Defn: A simplicial set is a functor $K : \Delta^{op} \to \mathrm{Sets}$.

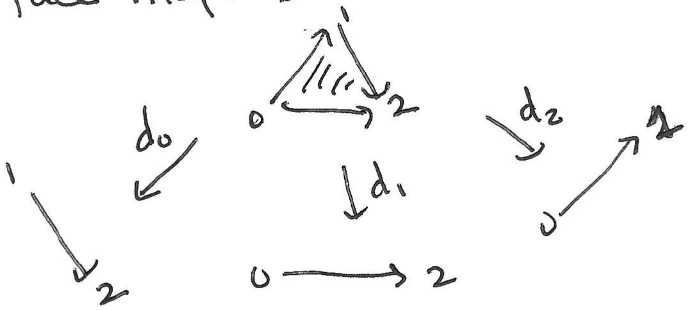
face maps

$$K_0 \leftleftarrows K_1 \quad K_2 \quad \cdots$$

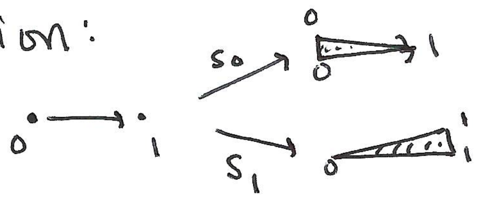
degeneracy maps

Idea: K_n is the set of "n-simplices"

face maps behave like those in simplicial complexes



degeneracy maps allow us to think of a simplex as one of a higher dimension:



Simplicial sets can be geometrically realized to topological spaces (essentially as depicted)

Ex: $\Delta[n]$ n-simplex

$\Delta[n]_k = \mathrm{Hom}_\Delta([k],[n])$ representable simplicial set.

geometrically realizes to the top'l n-simplex Δ^n.

Ex: Let G be a group. Its <u>nerve</u> is: nerve(G)

$$* \;\substack{\longleftarrow\\ \longleftarrow \\ \longrightarrow}\; G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow \\ \longrightarrow\\ \longrightarrow}\; G\times G \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow \\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G\times G\times G \;\cdots$$

Its geometric realization $|\mathrm{nerve}(G)|$ is BG, the classifying space of G, a $K(G,1)$.

More generally, for $\mathcal{C}$ a small category, its nerve is:

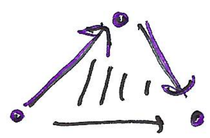
$$ob(\mathcal{C}) \;\substack{\xleftarrow{s}\\ \xleftarrow{t} \\ \xrightarrow[id]{}}\; mor(\mathcal{C}) \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow \\ \longrightarrow\\ \longrightarrow}\; mor(\mathcal{C}) \times_{ob(\mathcal{C})} mor(\mathcal{C}) \;\cdots$$

Defn: A <u>1-Segal</u> <u>set</u> is a simplicial set K such that

$$K_n \cong \underbrace{K_1 \times_{K_0} \cdots \times_{K_0} K_1}_{n} \qquad \forall n \geq 2.$$

Maps come from: $(\bullet \to \bullet \to \cdots \bullet \to \bullet) \hookrightarrow \Delta[n]$

$$\begin{gathered} (\bullet \to \bullet \to \cdots \bullet \to \bullet) \\ \shortparallel \\ \Delta[1] \amalg_{\Delta[0]} \cdots \amalg_{\Delta[0]} \Delta[1] \\ \shortparallel \\ G(n) \end{gathered}$$

e.g. $n=2$ $(\bullet \to \bullet \to \bullet) \hookrightarrow$ 

$$\mathrm{Hom}_{sSets}(\Delta[n], K) \longrightarrow \mathrm{Hom}_{sSets}(G(n), K) \qquad \text{"1-Segal maps"}$$
$$\| \qquad\qquad\qquad\qquad \|$$
$$K_n \qquad\qquad K_1 \times_{K_0} \cdots \times_{K_0} K_1$$

Note: 1-Segal sets K $\longleftrightarrow$ nerves of categories

K_0 = objects

K_1 = morphisms

$$\begin{tikzcd} K_1 \times_{K_0} K_1 \arrow[r, bend right, "\text{composition}"'] & K_2 \arrow[l, "(d_0, d_2)", "\cong"'] \arrow[r, "d_1"] & K_1 \end{tikzcd}$$

Can check: associativity

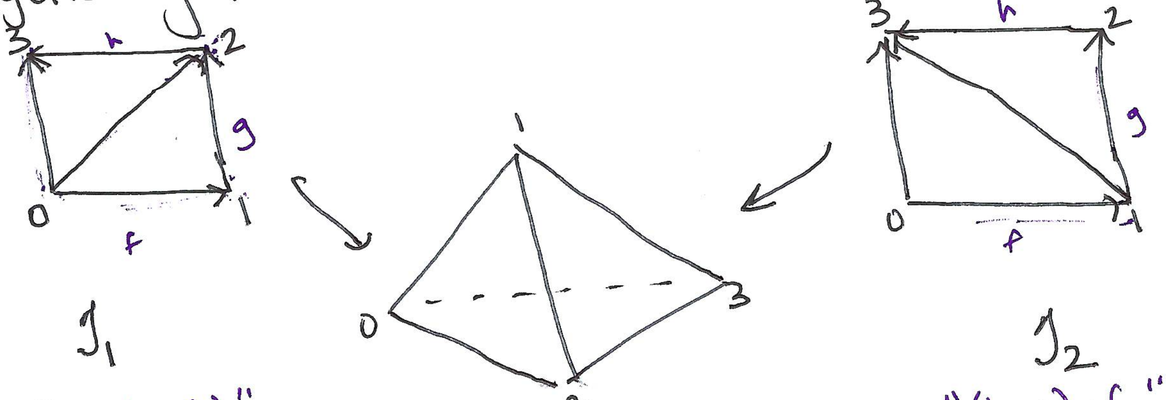
2-Segal sets: (Dyckerhoff-Kapranov)

(discrete decomposition spaces - Gálvez-Carrillo - Kock - Tonks)

Think of $G(n)$ as "triangulating" a segment $\bullet \to \bullet \to \cdot \cdot \cdot \to \bullet$

Move up a dimension: triangulate regular polygons with cyclically labeled vertices

e.g.



J_1 $\qquad\qquad\qquad\qquad$ J_2

"$h \circ (g \circ f)$" $\qquad\qquad$ "$(h \circ g) \circ f$"

Map into a simplicial set K:

$$\mathrm{Hom}(J_1, K) \xleftarrow{(d_1, d_3)} \mathrm{Hom}(\Delta[3], K) \xrightarrow{(d_0, d_2)} \mathrm{Hom}(J_2, K)$$
$$\cong \qquad\qquad\qquad \| \qquad\qquad\qquad \cong$$
$$K_2 \times_{K_1} K_2 \qquad\qquad K_3 \qquad\qquad K_2 \times_{K_1} K_2$$

Defn: A <u>2-Segal set</u> K is a simplicial set such that for $n \geq 3$ and every triangulation of a cyclically labelled regular $(n+1)$-gon by vertices, then the induced map $K_n \longrightarrow \underbrace{K_2 \times_{K_1} \cdots \times_{K_1} K_2}_{n-1}$ is an isomorphism.

"2-Segal maps"

What kind of structure is this?

still have "objects" K_0 and "morphisms" K_1.

try composition:
$$\begin{tikzcd} K_1 \times_{K_0} K_1 \arrow[rr, dashed, bend right, "?"'] & K_2 \arrow[l, "{(d_0,d_2)}"'] \arrow[r, "d_1"] & K_1 \end{tikzcd}$$

Best we can do: take preimage and apply d_1
could be empty or have multiple elements.

But, still associative.

Identity elements: degenerate 1-simplices

Dyckerhoff-Kapranov originally imposed a "unitality" condition, so identities composed as expected.

Thm: (Feller - Garner - Kock - Underhill-Proulx - Weber)
All 2-Segal sets are unital.

Example: partial monoids

set M together with $M_2 \subseteq M \times M$ "domain of multiplication"

$\cdot : M_2 \longrightarrow M$ satisfying $(m \cdot m') \cdot m''$ defined iff $m \cdot (m' \cdot m'')$ and they are equal

has a unit $1 \in M$ such that $(1, m), (m, 1) \in M_2$ for all $m \in M$

and $1 \cdot m = m \cdot 1 = m$

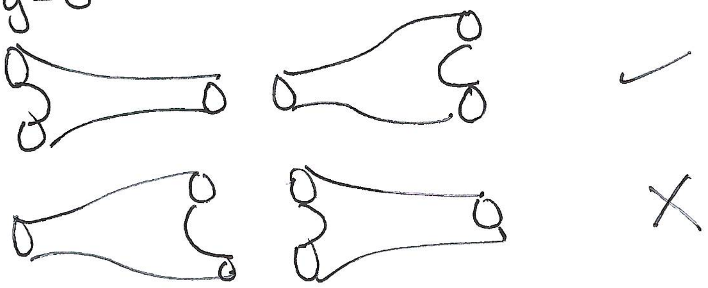
Can interpret as a simplicial set M:

$M_0 = \{1\}$

here composition is always defined.

If we impose a genus constraint, say $\leq g$, get a 2-Segal set $2\mathrm{Cob}^{\leq g}$

e.g. $g=0$



To come: 2-Segal spaces / objects

$S_\bullet$-construction

Hall algebras

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