Talk 2: 2-Segal spaces and the $S_\bullet$-construction

Today:
- 2-Segal spaces / 2-Segal objects
- How they arise from Waldhausen's $S_\bullet$-construction

Simplicial spaces: $X: \Delta^{op} \to SSets$ (or Top)

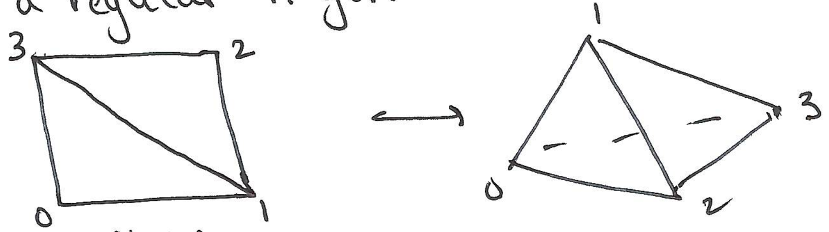
1-Segal spaces: $X_n \to \underbrace{X_1 \times^h_{X_0} \cdots \times^h_{X_0} X_1}_{n}$ weak equivalences $\forall n \geq 2$

$\rightsquigarrow$ simplicial / topological category up to homotopy
spaces of objects and morphisms

2-Segal spaces: $X_n \longrightarrow \underbrace{X_2 \times^h_{X_1} \cdots \times^h_{X_1} X_2}_{n-1}$ weak equivalences

$\forall n \geq 3$, triangulation of cyclically labeled triangulation of a regular n-gon.

e.g.,



$\rightsquigarrow X_3 \xrightarrow{(d_0, d_2)} X_2 \times_{X_1} X_2$

$\rightsquigarrow$ simp/top'l multivalued category up to homotopy.

Can generalize to simplicial objects in a sufficiently nice category, e.g., model category, $(\infty,1)$-category with limits.

Model structures:

Underlying category: simplicial spaces

injective ($=$ Reedy) model structure: levelwise weak equivs / cofibrations of spaces
(or projective)

left Bousfield localization with respect to:
- $\mathcal{G}(n) \hookrightarrow \Delta[n]$ $\forall n \geq 2$ $\rightsquigarrow$ fibrant objects 1-Segal
  $\to \cdots \to$
- $\mathcal{J} \longrightarrow \Delta[n]$ $\forall n \geq 3$, triangulation $\rightsquigarrow$ fibrant objects 2-Segal

Classical input for algebraic K-theory:
exact categories: "short exact sequences" make sense
(more general than abelian)
For $\mathcal{C}$ exact, define $K_0(\mathcal{C}) = \langle [C] \rangle / [C]=[B]+[D]$ for $0 \to B \hookrightarrow C \twoheadrightarrow D \to 0$

To define higher algebraic K-groups one approach uses the following features of an exact category: ("proto-exact category")

- pointed category: zero object $*$
- classes $\mathcal{M}$ of admissible monomorphisms ($\hookrightarrow$) containing $* \hookrightarrow A$ for all objects A, $\mathcal{E}$ of admissible epimorphisms ($\twoheadrightarrow$) containing $A \twoheadrightarrow *$ for all A, both closed under composition and containing all isomorphisms.
- a commutative square
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & B \arrow[d, two heads] \\ C \arrow[r, hook] & D \end{tikzcd}$$
is cartesian $\Leftrightarrow$ it is cocartesian (bicartesian)
- any
$$\begin{tikzcd} A \arrow[r, hook] \arrow[d, two heads] & B \\ C & \end{tikzcd}$$
and any
$$\begin{tikzcd} & B \arrow[d, two heads] \\ C \arrow[r, hook] & D \end{tikzcd}$$
can be completed to a bicartesian square as above, essentially uniquely.

Ex: R-modules with usual monomorphisms and epimorphisms

Ex: pointed sets (finite)
$\mathcal{M}$ = injections
$\mathcal{E}$ = surjections with singleton preimages over non-basepoints.

Waldhausen's $S_\bullet$-construction for a proto-exact category $\mathcal{C}$:
$S_0(\mathcal{C})$ = groupoid of zero objects of $\mathcal{C}$
$S_1(\mathcal{C})$ = groupoid of diagrams of the form
$$\begin{tikzcd} * \arrow[r, hook] & a \arrow[d, two heads] \\ & * \end{tikzcd}$$
(objects)

$S_2(\mathscr{C})$ = groupoid of diagrams
with square bicartesian
"short exact sequences"

$$\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[r, hook] \arrow[d, two heads] & a_{02} \arrow[d, two heads] \\
& * \arrow[r, hook] & a_{12} \arrow[d, two heads] \\
& & *
\end{tikzcd}$$

$S_n(\mathscr{C})$ = groupoid of diagrams
with squares bicartesian

$$\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[r, hook] \arrow[d, two heads] & a_{02} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & a_{0n} \arrow[d, two heads] \\
& * \arrow[r, hook] & a_{12} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & a_{1n} \arrow[d, two heads] \\
& & * \arrow[r, hook] & & \vdots \arrow[d, two heads] \\
& & & & *
\end{tikzcd}$$

Then $S_\cdot(\mathscr{C})$ is a simplicial groupoid.

- face maps d_i remove everything with index i

$$\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[r, hook] \arrow[d, two heads] & a_{02} \arrow[d, two heads] \\
& * \arrow[r, hook] & a_{12} \arrow[d, two heads] \\
& & *
\end{tikzcd}
\xrightarrow{d_0}
\begin{tikzcd}
* \arrow[r, hook] & a_{12} \arrow[d, two heads] \\
& *
\end{tikzcd}$$

$$\xrightarrow{d_1}
\begin{tikzcd}
* \arrow[r, hook] & a_{02} \arrow[d, two heads] \\
& *
\end{tikzcd}
\qquad
\xrightarrow{d_2}
\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[d, two heads] \\
& *
\end{tikzcd}$$

- degeneracy maps s_i repeat entries with identity maps

$$\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[d, two heads] \\
& *
\end{tikzcd}
\xrightarrow{s_0}
\begin{tikzcd}
* \arrow[r, equal] & * \arrow[r, hook] \arrow[d, equal] & a_{01} \arrow[d, equal] \\
& * \arrow[r, hook] & a_{01} \arrow[d, two heads] \\
& & *
\end{tikzcd}$$

$$\xrightarrow{s_1}
\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[r, equal] \arrow[d, two heads] & a_{01} \arrow[d, two heads] \\
& * \arrow[r, equal] & * \arrow[d, equal] \\
& & *
\end{tikzcd}$$

Taking the nerve levelwise gives a simplicial space (also $S_\bullet(\mathcal{C})$)
whose diagonal/geometric realization is a space $|S_\bullet(\mathcal{C})|$

Can check: $\pi_1 |S_\bullet(\mathcal{C})| \cong K_0(\mathcal{C})$

So, define the <u>K-theory space</u> of $\mathcal{C}$ to be $K(\mathcal{C}) = \Omega |S_\bullet(\mathcal{C})|$

and the K-groups $K_i(\mathcal{C}) = \pi_i K(\mathcal{C})$
$= \pi_{i+1} |S_\bullet(\mathcal{C})|$.

Thm: (Dyckerhoff-Kapranov, Gálvez-Carrillo-Kock-Tonks)
$S_\bullet(\mathcal{C})$ is a 2-Segal space.

Idea: $(d_0, d_2): S_3(\mathcal{C}) \longrightarrow S_2(\mathcal{C}) \times^h_{S_1(\mathcal{C})} S_2(\mathcal{C})$

$$\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[r, hook] \arrow[d, two heads] & a_{02} \arrow[r, hook] \arrow[d, two heads] & a_{03} \arrow[d, two heads] \\
& * \arrow[r, hook] & a_{12} \arrow[r, hook] \arrow[d, two heads] & a_{13} \arrow[d, two heads] \\
& & * \arrow[r, hook] & a_{23} \arrow[d, two heads] \\
& & & *
\end{tikzcd}
\qquad
\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[r, dashed] \arrow[d, two heads] \arrow[rr, bend left] & a_{02} \arrow[r, dashed] \arrow[d, dotted] & a_{03} \arrow[d, two heads] \\
& * \arrow[r, hook] \arrow[rr, bend left] & a_{12} \arrow[r, hook] \arrow[d, two heads] & a_{13} \arrow[d, two heads] \arrow[dd, bend left] \\
& & * \arrow[r, hook] & a_{23} \arrow[d, two heads] \\
& & & *
\end{tikzcd}$$

Not 1-Segal: $S_2(\mathcal{C}) \longrightarrow S_1(\mathcal{C}) \times^h_{S_0(\mathcal{C})} S_1(\mathcal{C})$

$$\begin{tikzcd}
* \arrow[r, hook] & a_{01} \arrow[r, dashed] \arrow[d, two heads] & \boxed{?} \arrow[d, dotted] \\
& * \arrow[r, hook] & a_{12} \arrow[d, two heads] \\
& & *
\end{tikzcd}$$

not all extensions split.

Note: Waldhausen originally defined this construction for categories with cofibrations
only have $\hookrightarrow$, pushouts, so couldn't necessarily complete above diagram
get "half" the structure: "left lower 2-Segal" (Tanner Carawan)

Q: What is the most general input for $S_\bullet$ for which the output is 2-Segal?

Let's look again at input data and see if we can generalize further

For simplicity: take $S_\bullet$ that is <u>discrete</u>: sets of diagrams of the above form $\rightsquigarrow$ $S_\bullet(\mathscr{C})$ is a simplicial set

- Need "horizontal" ($\hookrightarrow$) and "vertical" ($\twoheaddownarrow$) morphisms that assemble into squares

$$\begin{tikzcd} \bullet \arrow[r, hook] \arrow[d, two heads] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r, hook] & \bullet \end{tikzcd} \quad \rightsquigarrow \text{ \underline{double category}}$$

In particular, we don't need both kinds of arrows to be morphisms in a common ambient category.

- Since "bicartesian squares" no longer make sense in a double category, we require <u>stability</u>:

any
$$\begin{tikzcd} \bullet \arrow[r, hook] \arrow[d, two heads] & \bullet \\ \bullet & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & \bullet \arrow[d, two heads] \\ \bullet \arrow[r, hook] & \bullet \end{tikzcd}$$
uniquely determines
$$\begin{tikzcd} \bullet \arrow[r, hook] \arrow[d, two heads] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r, hook] & \bullet \end{tikzcd}.$$

- Analogue of a zero object: <u>pointed</u> double category
unique object $*$ that is initial in horizontal category
$$* \hookrightarrow a$$
and terminal in the vertical category
$$\begin{tikzcd} a \arrow[d, two heads] \\ * \end{tikzcd}.$$

$\rightsquigarrow$ <u>reduced</u> 2-Segal sets K : $K_0 = *$.

More general: <u>augmentation</u>
subset A of objects such that for any object x.
have unique $a \hookrightarrow x$,
$$\begin{tikzcd} x \arrow[d, two heads] \\ b \end{tikzcd}$$
with $a, b \in A$.

Thm: (B-Osorno-Ozornova-Rovelli-Scheimbauer)
The discrete $S_\bullet$-construction defines an equivalence of categories between augmented stable double categories and 2-Segal sets.

The inverse functor P takes a 2-Segal set K to the augmented double category PK with:

object set: K_1 augmentation: K_0

$\left.\begin{array}{l}\text{horizontal morphisms}\\ \text{vertical morphisms}\end{array}\right\} K_2$ (structure different)

squares K_3

Ex: M partial monoid $(a,b) \in M_2 \subseteq M \times M$

$\uparrow$ 2-simplex of the corresp. 2-Segal set.

vertical morphism:
$$\begin{tikzcd} a\cdot b \arrow[d, "a\cdot"'] \\ b \end{tikzcd}$$

horizontal morphism: $a \xrightarrow{\cdot b} a\cdot b$

$(a,b,c) \in M_3$ corresponds to
$$\begin{tikzcd} a\cdot b \arrow[r, "\cdot c"] \arrow[d, "a\cdot"'] & (a\cdot b)\cdot c = a\cdot(b\cdot c) \arrow[d, "a\cdot"] \\ b \arrow[r, "\cdot c"] & b\cdot c \end{tikzcd}$$

associativity gives stability