Talk 3: The general $S_\bullet$-construction and Hall algebras

Last time: • Introduced the $S_\bullet$-construction a (proto-)exact category, which is 2-Segal.
• Talked about generalizing the input for a discrete version.

Today: • Back to generalizing the original version.
• Hall algebras from 2-Segal objects

Recall: $S_n(\mathcal{C})$ is the groupoid of diagrams

$$\begin{tikzcd} * \arrow[r, hook] & a_{01} \arrow[r, hook] \arrow[d, two heads] & a_{02} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & a_{0n} \arrow[d, two heads] \\ & * \arrow[r, hook] & a_{12} \arrow[r, hook] \arrow[d, two heads] & \cdots \arrow[r, hook] & a_{1n} \\ & & * \cdots & & \vdots \arrow[d, two heads] \\ & & & & * \end{tikzcd}$$

Discrete ⇝ homotopy version
• double category ⇝ double (1-)Segal space
$X: \Delta^{op} \times \Delta^{op} \to sSets$ that is a Segal space in each variable

$X([1],[1])$

"double category up to homotopy"

• stability: $\begin{tikzcd} \bullet \arrow[r] \arrow[d, two heads] & \bullet \\ \bullet & \end{tikzcd}$ or $\begin{tikzcd} & \bullet \arrow[d, two heads] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$ uniquely determines a square $\begin{tikzcd} \bullet \arrow[r] \arrow[d, two heads] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r] & \bullet \end{tikzcd}$.

⇝ now determines a square up to homotopy / up to a contractible space of choices.

• pointed: object $*$ that is initial for the horizontal category and terminal for the vertical category
⇝ space of such objects, space of maps to/from other objects are contractible.

~~Simp~~ similarly for augmentation

Organize this data: category Σ which is given by adjoining a terminal object $[-1]$ to $\Delta \times \Delta$.

Look at functors $\Sigma^{op} \to sSets$

Thm: (B.-Osorno-Ozornova-Rovelli-Scheimbauer)

a) There is a model structure on $sSets^{\Sigma^{op}}$ with fibrant objects the augmented stable double Segal spaces.

b) The $S_\bullet$-construction defines the right adjoint of a Quillen equivalence between it and the 2-Segal space model structure.

c) On (proto-)exact (quasi-)categories, this $S_\bullet$ recovers classical/known versions.

Related work: CGW categories and variants (Campbell-Zakharevich, Sarazola-Shapiro)

also general input for K-theory construction

Work in progress: (Shapiro and Zakharevich)

- CGW categories can be understood as pointed stable double Segal spaces
- identifying which ones they are

Hall algebras

Let $\mathcal{A}$ be an abelian category with $|\mathrm{Hom}(A,B)|, |\mathrm{Ext}^1(A,B)| < \infty$ for all objects A, B. (finitary)

Defn: Its Hall algebra $\mathcal{H}(\mathcal{A})$ is defined by:

- vector space (over some field $\Bbbk$) with basis isomorphism classes of objects in $\mathcal{A}$.
- multiplication: $[A]\cdot[B] = \sum_{[C]} g^C_{AB}[C]$, where

$$g^C_{AB} = \frac{|\{0 \to A \to C \to B \to 0\}|}{|\mathrm{Aut}(A)|\,|\mathrm{Aut}(B)|}.$$

Defines an associative, unital algebra.

Ex: Let $\mathfrak{g}$ be a Lie algebra of type A, D, E.

It has an associated simply laced Dynkin diagram, e.g. A_3 $\bullet - \bullet - \bullet$

Add orientations of edges to get a quiver Q, e.g., $\bullet \rightarrow \bullet \leftarrow \bullet$

Let $\mathcal{A}$ be the category of $\mathbb{F}_q$-representations of Q.

e.g., $\mathbb{F}_q^3 \longrightarrow \mathbb{F}_q^2 \longleftarrow \mathbb{F}_q$

Forms a finitary abelian category $\rightsquigarrow$ Hall algebra $\mathcal{H}(Q)$.

Has a close relation with the quantum group $\mathcal{U}_q(\mathfrak{g})$ (Ringel, Green)

Reinterpretation: Let $\mathcal{A}$ be a finitary abelian category.

Then $S_\bullet(\mathcal{A})$ is a 2-Segal groupoid / space.

$S_0(\mathcal{A})$ groupoid of zero objects

$S_1(\mathcal{A})$ groupoid with components corresponding to isomorphism classes of objects

$S_2(\mathcal{A})$ groupoid of short exact sequences, etc.

Recall that "composition" is defined via the span

$$S_1(\mathcal{A}) \times S_1(\mathcal{A}) \xleftarrow{(d_2,d_0)} S_2(\mathcal{A}) \xrightarrow{d_1} S_1(\mathcal{A})$$

$$\cancel{(B,A)}\ (A,B) \longleftarrow\!\shortmid (0\to A\to C\to B\to 0) \longmapsto C$$

by assumption, this map has finite fibers.

For a field $\Bbbk$, take finitely-supported functions $\pi_0 S_1(\mathcal{A}) \to \Bbbk$.

$\rightsquigarrow$ defines a $\Bbbk$-vector space $\Bbbk^{\pi_0 S_1(\mathcal{A})}$.

Define a multiplication on it via:

$$\Bbbk^{\pi_0 S_1(\mathcal{A})} \times \Bbbk^{\pi_0 S_1(\mathcal{A})} \cong \Bbbk^{\pi_0 S_1(\mathcal{A}) \times \pi_0 S_1(\mathcal{A})} \xrightarrow{(d_2,d_0)^*} \Bbbk^{\pi_0 S_2(\mathcal{A})} \xrightarrow{(d_1)_!} \Bbbk^{\pi_0 S_1(\mathcal{A})}.$$

$$f \longmapsto \Big([C] \longmapsto \underbrace{\sum_{[S],\, d_1(S)=C}}_{\text{short exact sequences}} \underbrace{f(d_2,d_0)(S)}_{=f(A,B)}\Big)$$

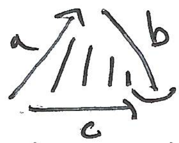
If $f = (\mathbb{1}_{[A]}, \mathbb{1}_{[B]})$, we get $\displaystyle\sum_{[C]} \sum_{[0\to A\to C\to B\to 0]} \mathbb{1}_{[C]}$

$$= \sum_{[C]} g^C_{AB}\, \mathbb{1}_{[C]},$$ recovering the definition above.

This same kind of construction holds for sufficiently finitary 2-Segal objects X with $X_0 = *$.
(otherwise, get a $\mathbb{k}$-linear category with object set X_0).

Defn: Let K be a reduced 2-Segal set, and let $\mathbb{k}$ be a field. The Hall algebra $\mathcal{H}(K) (= \mathcal{H}_{\mathbb{k}}(K))$ is defined as:

- the $\mathbb{k}$-vector space of finitely supported functions $K_1 \to \mathbb{k}$.
- multiplication given by $\mathbb{1}_a * \mathbb{1}_b = \sum_c g_{ab}^c \mathbb{1}_c$, where g_{ab}^c is the number of 2-simplices $z \in K_2$ such that $d_0(z) = b$, $d_1(z) = c$, $d_2(z) = a$.



Notes:
- Can recover above classical example using isomorphism classes in 2-Segal groupoids.
- Can recover variants like derived/motivic Hall algebras (Dyckerhoff-Kapranov)
- Can retain more structure to get a Hall $(\infty,2)$-category.

Ex: Let M be a partial monoid. Then $\mathcal{H}(M)$ is the algebra:

- vector space spanned by elements of M.
- $\mathbb{1}_m * \mathbb{1}_n = \begin{cases} m \cdot n & \text{if } m, n \in M_2 \\ 0 & \text{otherwise} \end{cases}$

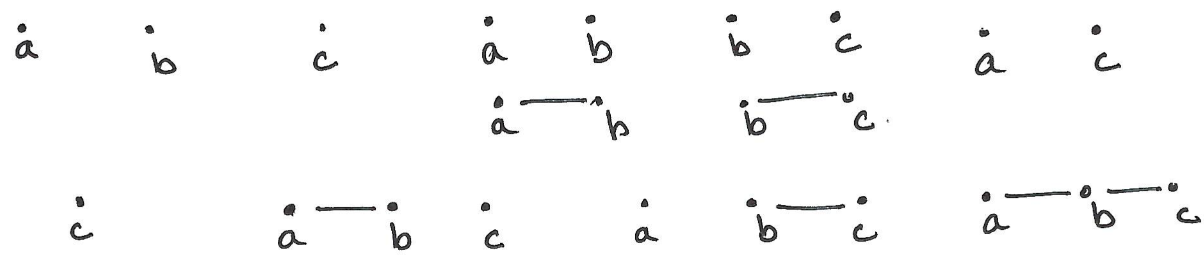
In this case, composition is unique when it exists, so there are no longer sums or interesting coefficients.

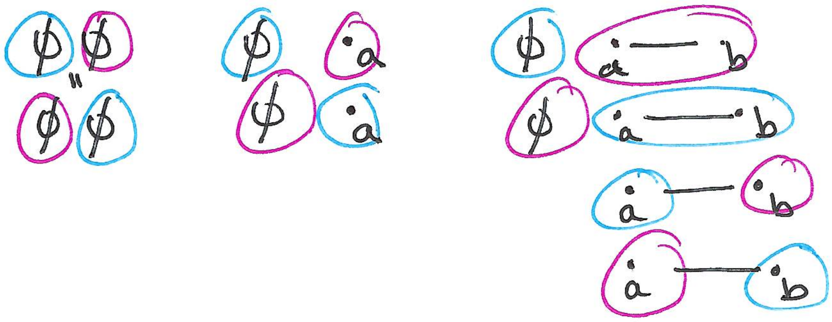
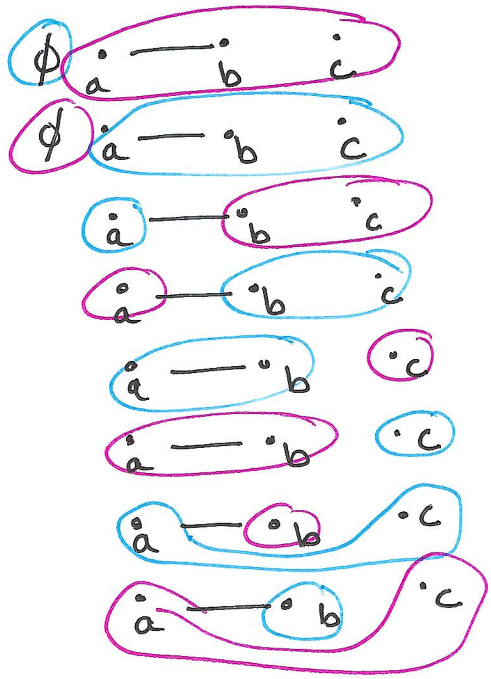
Ex: Let G be a finite graph.
Define an associated simplicial set $X (= X_G)$

$X_0 = \{\emptyset\}$
$X_1 = \{\text{subgraphs of } G\}$
$X_n = \{\text{subgraphs } H \text{ of } G \text{ equipped with an ordered partition of the vertices into } n \text{ possibly empty subsets}\}$.

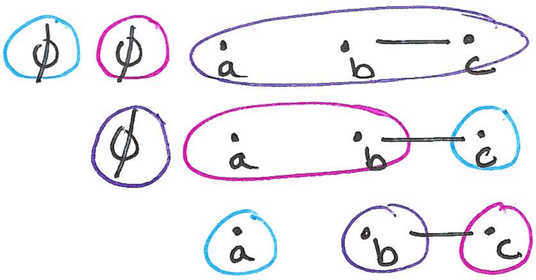
Let G be 

X_0: $\emptyset$

X_1: $\emptyset$ 

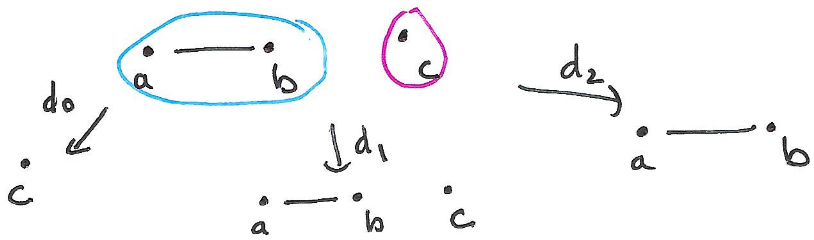
X_2:  

blue < pink < purple

X_3: 

etc.

face maps: $X_2 \to X_1$



degeneracy maps $X_1 \to X_2$