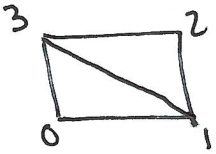
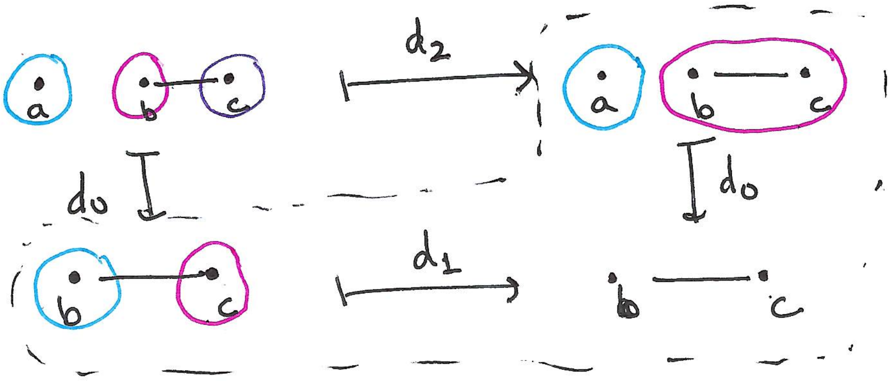
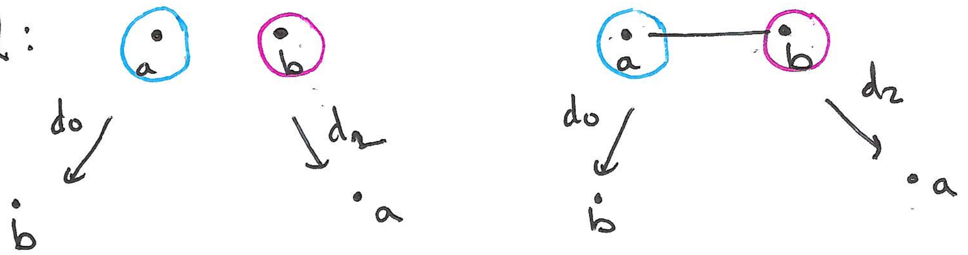
Talk 4: The conclusion?

Back to graph example...

2-Segal: for map associated to 

$$\begin{tikzcd} X_3 \arrow[r, "d_2"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_2 \arrow[d, "d_0"] \\ X_2 \arrow[r, "d_1"'] & X_1 \end{tikzcd}$$

e.g. 

Not 1-Segal: 

This example shows $X_2 \xrightarrow{(d_2, d_0)} X_1 \times X_1$ is not injective

Also not surjective: for example, can't "compose"

$\dot{a}$ $\dot{b}$ with b — c

In fact, can only "compose" disjoint subgraphs.

Apply the Hall algebra construction:

- $X_0 = *$, so do get an algebra
- vector space spanned by elements of X_1, i.e, subgraphs of G.
- multiplication: $H * K = \sum_J g_{HK}^J \, J$, where $H, K, J \subseteq G$, and g_{HK}^J counts 2-simplices

$$\begin{tikzcd} & \cdot \arrow[dr, "K"] & \\ \cdot \arrow[ur, "H"] \arrow[rr, "J"'] & & \cdot \end{tikzcd}$$

i.e., subgraphs J with a partition of the vertices into 2 subsets such that H, K are full subgraphs on those 2 sets that necessarily disjoint.

• ϕ is the unit element (always the case)

Ex: $G = (\overset{\bullet}{a} \text{——} \overset{\bullet}{b})$

algebra is 5-dim'l: $[\phi]$, $[a]$, $[b]$, $[\overset{\bullet}{a} \quad \overset{\bullet}{b}]$, $[\overset{\bullet}{a} \text{——} \overset{\bullet}{b}]$

The only nontrivial multiplication is

$$[a] * [b] = [\overset{\bullet}{a} \quad \overset{\bullet}{b}] + [\overset{\bullet}{a} \text{——} \overset{\bullet}{b}] = [b] * [a]$$

Ex: $G = (\overset{\bullet}{a}\circlearrowleft)$

subgraphs are $[\phi]$, $[\overset{\bullet}{a}]$, $[\overset{\bullet}{a}\circlearrowleft]$

There is no nontrivial multiplication, since the two nonempty subgraphs are not disjoint.

Properties of $\mathcal{H}(X_G)$:

1) If $H \cap K \neq \phi$, then $H * K = 0$.
(no 2-simplex with $d_0 = K$ and $d_2 = H$)

2) If $a \neq b$ are vertices of G, then $a * b = \sum_H H$ where $v(H) = \{a, b\}$

e.g., $a * b = [\overset{\bullet}{a} \quad \overset{\bullet}{b}] + [\overset{\bullet}{a} \text{——} \overset{\bullet}{b}]$ above.

3) More generally, if $H \cap K = \phi$, then $H * K = \sum_J J$ where $H \amalg K \subseteq J$, $v(J) = v(H) \amalg v(K)$.

e.g., for $G = (\overset{\bullet}{a} \text{——} \overset{\bullet}{b} \overset{f}{\underset{g}{=\!=}} \overset{\bullet}{c})$ if $H = [\overset{\bullet}{a} \text{——} \overset{\bullet}{b}]$, $K = [\overset{\bullet}{c}]$, then

$H * K = [\overset{\bullet}{a} - \overset{\bullet}{b} \quad \overset{\bullet}{c}] + [\overset{\bullet}{a} - \overset{\bullet}{b} \overset{f}{\text{——}} \overset{\bullet}{c}] + [\overset{\bullet}{a} - \overset{\bullet}{b} \overset{g}{\text{——}} \overset{\bullet}{c}] + [\overset{\bullet}{a} \text{—} \overset{\bullet}{b} =$

4) $\mathcal{H}(X_G)$ is commutative
partitions come in pairs that are invisible to $\mathcal{H}$.

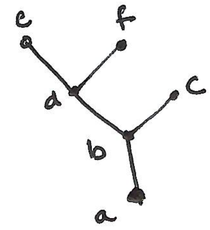
5) The coefficients g^J_{HK} are always either 0 or 1.

$a \bullet \text{——} \bullet_b \circlearrowleft$

$[\overset{\bullet}{a}] * [\overset{\bullet}{b}\circlearrowleft] = [\overset{\bullet}{a} \quad \overset{\bullet}{b}\circlearrowleft] + [\overset{\bullet}{a} \text{——} \bullet\circlearrowleft_b]$

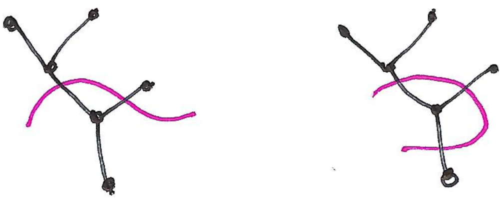
$[\overset{\bullet}{a}] * [\overset{\bullet}{b}] = [\overset{\bullet}{a} \quad \overset{\bullet}{b}] + [\overset{\bullet}{a} \text{——} \overset{\bullet}{b}]$

Variant: 2-Segal sets associated to rooted trees T

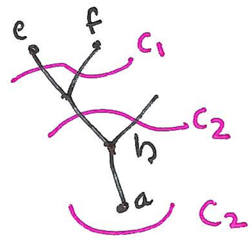


(joint w/ Borghi, Dey, Gálvez-Carrillo, Hoekstra-Mendoza)

Admissible cuts: divide T into a lower subtree, either containing the root or empty) and an upper tree/forest



Can have multiple compatible cuts:



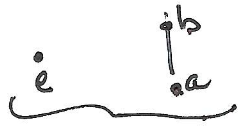
Define a simplicial set Y ($=Y_T$):

$Y_0 = *$

Y_1 is the collection of all subforests of T obtained from admissible cuts

e.g.

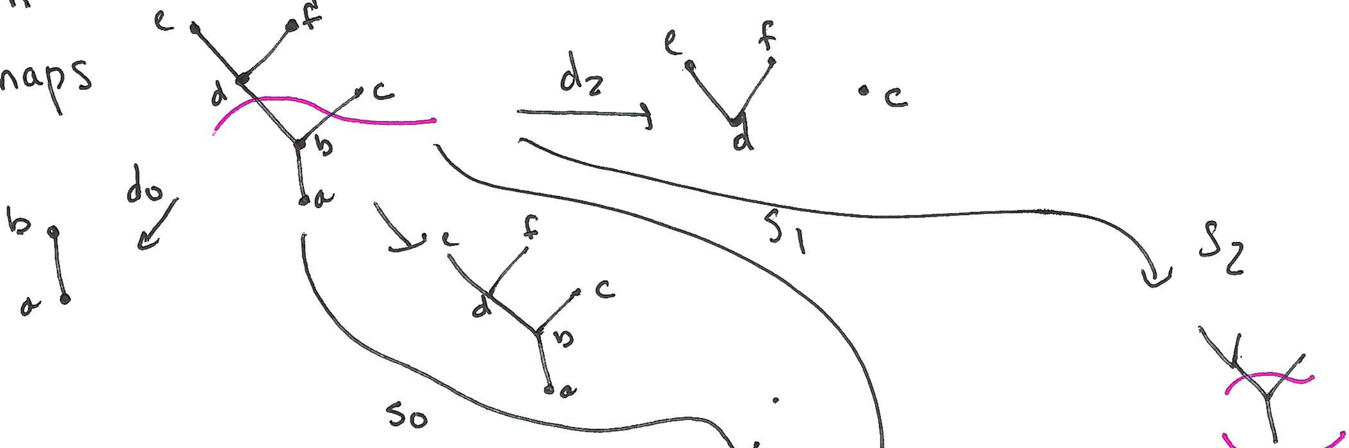
(from c_1)



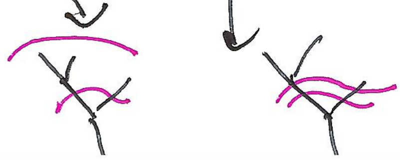
Y_2 consists of elements of Y_1 together with an admissible cut (requires cuts for forests)

Y_n consists of element of Y_1 with $(n-1)$ admissible cuts.

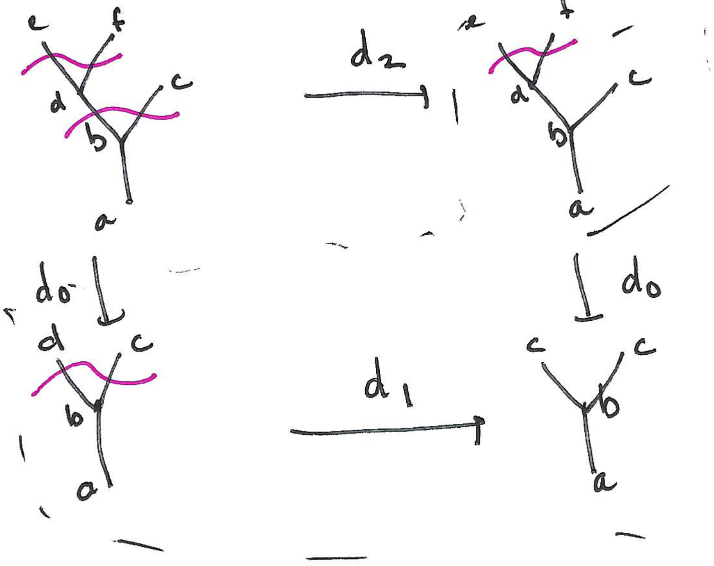
Face maps



Degeneracy maps insert cuts



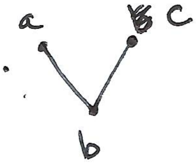
Y_T is 2-Segal, e.g.,



Not 1-Segal, since can't always compose.

Prop: Given a tree T, Y_T forms a sub-2-Segal set of X_T, where we consider the underlying subgraph of T.

Ex:  $Y_1 = \{\phi, a, b, c, b - a, c - b, c - b - a\} \subsetneq X_1$

vs.  $Y_1 = \{\phi, a, b, c, a - b, b - c, a\ c, a - b - c\}$

Can also look at the Hall algebras, but since this construction is not functorial, we don't get an algebra map
$\mathcal{H}(Y_T) \to \mathcal{H}(X_T)$.

$\mathcal{H}(Y_T)$ is not commutative, due to the "ordering" on the tree

 $[\cdot_b] * [\cdot_a] = [{}^{b}_{a}]$

$[\cdot_a] * [\cdot_b] = 0$

Toward higher Segal spaces

Prop: (Dyckerhoff Kapranov)
$X: \Delta^{op} \to SSets$ is 2-Segal iff the 2-Segal maps corresponding to triangulations  (a) and  (b) are weak equivs $\forall n \geq 3$.

Defn: A simp. space is:

- <u>lower 2-Segal</u> if the maps for (a) triangulations are wk equivs $\forall n \geq 3$
- <u>upper 2-Segal</u> if the maps for (b) triangulations are wk equivs $\forall n \geq 3$.

Cor: X is 2-Segal $\Leftrightarrow$ it is both upper and lower 2-Segal.

For $d \geq 3$, can generalize these definitions to ones for upper and lower d-Segal spaces, using cyclic polytopes.

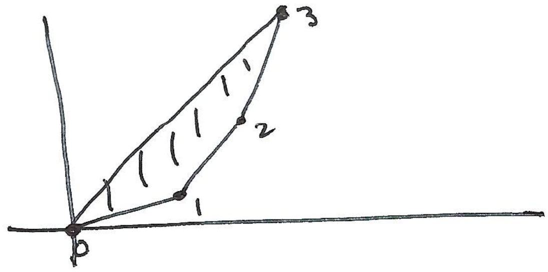
Consider
$$v: \mathbb{R} \to \mathbb{R}^d$$
$$t \mapsto (t, t^2, \dots, t^d)$$

Defn: The <u>cyclic</u> <u>d-dim'l</u> <u>polytope</u> on $[n]$ is the convex hull of $v([n])$, denoted by $C([n], d)$.

Ex: $C([3], 0) = \{0\}$ $\qquad v: t \mapsto 0$

$C([3], 1) = (\bullet_0 - \bullet_1 - \bullet_2 - \bullet_3)$ $\qquad v: t \mapsto t$

$C([3], 2)$ $\qquad v: t \mapsto (t, t^2)$



$C([3], 3) \cong \Delta^3$.

There is a combinatorial description of "upper" and "lower" subsets of $[n]$ that can be used to define upper and lower d-Segal-maps, recovering the above notions when $d = 2$.

- n-excisive functors (Walde)
- higher ~~two~~ $S_\bullet$-constructions? (see Poguntke)
- algebraic structure (Gal-Gal)