eCHT Stable Homotopy Theory Course

Instructor: **Kiran Luecke**
Notes Typed by **Mattie Ji**

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These are lecture notes for the **eCHT** (electronic Computational Homotopy Theory) **Stable Homotopy Theory Course** in the **Winter 2026** semester, with **Kiran Luecke** as the lead instructor. The graduate teaching assistants are **Mattie Ji** and **Jesse Keyes**. This course is based on the syllabus **David Mehrle** designed for the previous iteration of the course in Winter 2025.

This course is part of the broader eCHT program, led by principal investigator (PI) **Dan Isaksen**, with co-PIs **Bert Guillou**, **J. D. Quigley**, and **Vesna Stojanoska**. The eCHT stable homotopy theory class is supported by the **eCHT Research Training Group in the Mathematical Sciences grant** from the **National Science Foundation**.

These notes are typed by Mattie Ji. If you find any mistakes in these notes, please feel free to direct them to the notetaker (Mattie Ji).

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Overview

The **electronic Computational Homotopy Theory (eCHT)** is an online research community supporting a variety of mathematical activities mostly in algebraic topology, homotopy theory, and related topics. The **eCHT Stable Homotopy Theory Course** (Winter 2026) is an online course introducing **stable homotopy theory** from the point-set perspective of spectra. This course is based on the syllabus David Mehrle designed for the previous iteration of the course in Winter 2025.

This document contains the lecture notes for the class and an appendix on additional materials related to the class typed by Mattie Ji. These notes, with the exception of sections labeled with $*$, were originally live-texed by Mattie Ji, with additional editing and expansions written afterwards. The sections labeled with $*$ were written by Mattie Ji after the original talks had concluded. The appendix is written by Mattie Ji.

For more details, please visit the course website at

https://s.wayne.edu/echt/echt-courses/stable-homotopy-theory-winter-2026/

and the new website at

https://echtorc.github.io/course-on-stable-homotopy-theory/.

The notetaker would like to thank Dan Isaksen, Jesse Keyes, Kiran Luecke, David Mehrle, and all the participants in the course for their support, contributions, and helpful input throughout the semester. The notetaker thanks Pengkun Huang for miscellaneous helpful conversations in the writing of this note. The notetaker would also like to thank Mark Behrens for helpful conversations and comments on selected portions of these notes, and Jeremy Hahn for the teaching the notetaker about the materials presented in Appendix B.1.

The notetaker would also like to thank Ado (アド), whose music and tremendous encouragement helped the notetaker to finish the writing of this document.

Suggestions on How to Read This Document

This course assumes a pre-requisites in:

1. The equivalent of two semesters of a first course in **algebraic topology**, including homology, cohomology, and homotopy groups π_n for $n \geq 1$. Some knowledge of obstruction theory (e.g., obstruction theory) is also helpful, but they are partially recalled in Week 7.

2. Working knowledge of **category theory**, including: categories, functors, natural transformations, limits, colimits, adjunctions. Knowledge of abelian and symmetric categories are helpful but will be covered in the notes.

David Mehrle's original syllabus detailed a dependency tree and reading order for the lectures outlined. For convenience, we also outline it here. Since the actual course deviated slightly from the syllabus, a slight adaptation of the order is as follows. Note that Week 1 of this course is a thematic introduction to the course, and the main lectures begin on Week 2.

$$\begin{tikzcd}
\text{(2) Categorical Homotopy Theory} \arrow[r] \arrow[ddddr, bend right] & \text{(2.1.5) Homotopy Colimits (in Spaces)} \arrow[d] & \text{(3) Stability and Spaces} \arrow[dl] \\
& \text{(5) Sequential Spectra} \arrow[d] & \text{(4) Generalized Cohomologies} \arrow[l] \\
& \text{(6) Stability Theorems} \arrow[d] & \text{Appendix (A)} \arrow[l] \\
& \text{(7) Cellular Spectra} \arrow[d] & \\
& \text{(8) Stable Homotopy Category} \arrow[d] & \\
\text{(10) Symmetric and Orthogonal Spectra} \arrow[d] & \text{(9) Smash Product} \arrow[l] \arrow[r] \arrow[d] & \text{(13) Localization} \arrow[d] \\
\text{(12) Operads} \arrow[dr] & \text{(11) Ring Spectra} \arrow[l] \arrow[d] & \text{(14) Chromatic Homotopy} \arrow[dl] \\
& \text{(B) Chromatic } \mathbb{E}_n\text{-Structures} &
\end{tikzcd}$$

In particular, the material originally designated as "Week 1 on **Homotopical Categories**" and "Week 2 on **Homotopy (Co)Limits**" in the syllabus has been consolidated into a single week for Week 2. The corresponding section has been expanded to incorporate the topics indicated in the original syllabus. Week 2 provides an introduction to categorical homotopy theory, following [Rie14].

For much of the subsequent material, however, it is enough to be familiar with what a homotopical/homotopy category is (Section 2.1.1) and what homotopy (co)limits of spaces are for several standard classes of diagrams; these examples are reviewed in Section 2.1.5. Much of the discussions on deformations and derived functors are used in Week 8 and 9.

1 Week 1: Course Introduction by Kiran Luecke

1.1 Lecture January 20th, 2026 by Kiran Luecke

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1.1.1 Logisitics

In the first meeting, we will discuss some logistics for this class. There will be two meetings a week on Tuesdays and Thursdays from 1 - 2 pm EST.

- On Tuesdays, we will be having student presentations on the topic.
- On Thursdays, we will be having problem sessions on problems designed to accompany the topics.

1.1.2 What is Stable Homotopy Theory

The **modern formulation** of the subject is that it is the study of stable (presentable) ∞-categories. On the other hand, the materials discussed for this course will contain none of that. This is a course on stable homotopy theory from the point-set perspective of spectra.

We will take a more **classic formulation of the field**, that the subject is the study of stable phenomena in spaces.

Definition 1.1. We say a map $f : X \to Y$ is a **stable weak equivalence** if for all n, there exists $k >> 0$, such that the induced map

$$(\Sigma^k f)_* : \pi_n(\Sigma^k X) \to \pi_n(\Sigma^k Y) \text{ is an isomorphism.}$$

We will discuss this in more details later, but considering topological spaces equipped with **stable weak equivalences** is often considered as living inside what is called $\mathrm{ho}(\mathrm{Sp})$, the homotopy category of spectra. Informally, the category of spectra Sp is a place to study stable phenonmena arising from spaces.

Remark 1.2. Although we have not defined spectra either, a more modern perspective on spectra is that Sp is the free stable ∞-category on 1 object. Again, there are no ∞-categories in the syllabus, but the spreaker can briefly touch on them if they wish.

So what are some key features of a **stable category**? Thematically and informally, they are characterized by the following:

1. Finite products = Finite sums. In particular, this equips every object with the "structure of an abelian group" (abelian group objects) since we have a natural map

$$A \times A = A \bigsqcup A \to A.$$

   In the case where we have product of 0 things and sum of 0 things, this just means initial and terminal objects agree, and we have a zero object. A common slogan here is that Sp is "the ∞-categorical version of abelian groups".

2. In fact, finite limits and colimits agree. To assert this, it is enough to say that pushouts and pullbacks are the same, ie. the following square is a pushout if and only if it is a pullback.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

As a particular consequence, if we take Y and Z to be points (or zero objects), we can define suspension Σ and loop Ω as

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} \Omega W \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & W \end{tikzcd}$$

3. We also have "cofiber sequences" of the form

$$X \xrightarrow{f} Y \to Cf \to \Sigma X \to \Sigma Y \to \ldots$$

4. The category Sp is symmetric monoidal. The symmetric monoidal operation is classically called the "smash product" $X \wedge Y$. We sometimes want to write it as $X \otimes Y$ instead.

5. The initial ring object in Sp is called the **sphere spectrum** (denoted $\mathbb{S}$).

6. There is an important adjunction
$$\text{Spaces} \;{}_{\Omega^\infty}\!\rightleftarrows^{\Sigma^\infty_+}\; \mathrm{Sp}\,.$$
Here one can think of $\Omega^\infty \mathbb{S} := \operatorname{colim}_n \Omega^n S^n$ and Σ^∞_+ is the **suspension spectra operation**. We will discuss them more precisely later in the course.

7. From (5), it turns out we should really think of $\mathrm{Sp} = \mathrm{Mod}(\mathbb{S})$ (i.e. modules over the sphere spectrum). Moreover, in Sp, there is a map of rings

$$\mathbb{S} \to \mathbb{Z} \text{ which is a truncation to } \pi_0\text{-homotopy groups.}$$

To be precise, it turns out

$$\pi_0(\mathbb{S}) = \mathbb{Z}, \pi_1(\mathbb{S}) = \pi_2(\mathbb{S}) = \mathbb{Z}/2, \pi_3(\mathbb{S}) = \mathbb{Z}/24, \ldots$$

and we can think of $\mathbb{Z}$ as
$$\pi_0(\mathbb{Z}) = \mathbb{Z}, \pi_{>0}(\mathbb{Z}) = 0.$$
Thus the map is really a truncation of homotopy groups. We will also see later that we have maps

$$\mathrm{Sp} = \mathrm{Mod}(\mathbb{S}) \rightleftarrows \mathrm{Ch}(\mathbb{Z}) = \mathrm{Mod}(\mathbb{Z})$$

Here the reason why we write chain complexes over $\mathbb{Z}$ is because we want to think in a "derived sense". This map in particular associated
$$\mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2 \to \mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2,$$
where the LHS should be thought of as a refinement. Here one has that

$$\pi_0(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2) = \mathbb{Z}/2, \pi_1(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2) = \mathbb{Z}/2, \pi_2(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2) = \mathbb{Z}/2, \pi_3(\mathbb{F}_2 \otimes_{\mathbb{S}} \mathbb{F}_2) = (\mathbb{Z}/2)^2.$$

The homotopy groups of $\mathbb{F}_2 \otimes_{\mathbb{Z}} \mathbb{F}_2$ has a relatively nice description in terms of Tor.

8. Sp is the home of generalized cohomology theories. Namely, given a **generalized cohomology theory**

$$\mathrm{Top} \to \mathrm{gr\,Ab}, X \mapsto E^* X,$$

there exists a spectrum $E \in \mathrm{Sp}$ such that

$$E^* X = \pi_{-*} \mathrm{Map}_{\mathrm{Sp}}(\Sigma^\infty_+ X, E).$$

Example 1.3. If we specify $E^* X = H^*(X; \mathbb{Z})$, then we just take $E = \mathbb{Z}$. Note that $\Omega^\infty \mathbb{Z}$ is just $\mathbb{Z} = K(\mathbb{Z}, 0)$, considered as a topological space. You can argue there is some abuse of notations here.

When we consider $\mathbb{Z}$ as a spectrum, one sometimes use $H\mathbb{Z}$ to make the distinctions.

If we specify $E^*X = H^*(X; \mathbb{F}_2)$, then we just take $E = \mathbb{F}_2$.

If we take E^*X to be complex K-theory, then E is represented as some spectrum called KU where $\Omega^\infty \mathrm{KU} = \mathbb{Z} \times \mathrm{BU}$.

In this course, we will see how the themes mentioned above show up in stable homotopy theory, leading to the construction and discussions for the various models for spectra.

Here is a question/exercise for the participants.

Question 1.4. What is $\Omega^\infty\Sigma\mathbb{Z}$? **Hint:** Pushouts agree with pullbacks, $\Sigma\mathbb{Z}$ is defined by pushouts, Ω^∞ is a right-adjoint that commutes with limits.

Solution: Combining the discussions above, we have a pullback square on the right

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \bullet \arrow[d] \\ \bullet \arrow[r] & \Sigma\mathbb{Z} \end{tikzcd} \quad \to \quad \begin{tikzcd} \mathbb{Z} = \Omega^\infty\mathbb{Z} \arrow[r] \arrow[d] & \Omega^\infty\bullet = \bullet \arrow[d] \\ \Omega^\infty\bullet = \bullet \arrow[r] & \Omega^\infty\Sigma\mathbb{Z} \end{tikzcd}.$$

Thus we have that $\mathbb{Z} = \Omega(\Omega^\infty\Sigma\mathbb{Z})$. Thus, $\Omega^\infty\Sigma\mathbb{Z} = S^1$ as the loop space of S^1 is equivalent to $\mathbb{Z}$. Note that $K(\mathbb{Z}, 1) = S^1$. In general, we also have that

$$\Omega^\infty\Sigma^n\mathbb{Z} = K(\mathbb{Z}, n).$$

1.2 Problem Sessions Addendum by Kiran Luecke

In this problem session, we will be discussing some problems that are counted as "pre-requisite materials".

Question 1.5. Describe the 0th-through-3rd layers of the Postnikov Tower of S^2.

We wish to first recall what Postnikov towers are.

Definition 1.6. Let X be path-connected, a **Postnikov tower** is a tower of fibrations

$$\begin{tikzcd} & \vdots \arrow[d] \\ & P_3(X) \arrow[d] \\ & P_2(X) \arrow[d] \\ & P_1(X) \arrow[d] \\ X \arrow[r, dashed, "f_0"] \arrow[ur, "f_1"] \arrow[uur, "f_2"] \arrow[uuur, "f_3"] \arrow[uuuur] & P_0(X) = * \end{tikzcd}$$

such that:

1. $\pi_k(P_n(X)) = 0$ for $k > n$.

2. $\pi_k(X) \to \pi_k(P_n(X))$ is an isomorphism for $k \leq n$.
3. Let F_n be the fiber of $P_n(X) \to P_{n-1}(X)$, then $F_n = K(\pi_n(X), n)$.

Remark 1.7. There is a way to construct Postnikov towers using localizations, which may or may not be touched on in one of the later lectures about Postnikov towers.

Let us see what the 0-th to 3rd stage of a Postnikov tower of S^2 could be. Evidently, the 0th, 1st, 2nd stage of the tower can be given by

$$\begin{tikzcd}
& K(\mathbb{Z},2) \arrow[d] \\
& * = K(0,1) \times K(0,,0) \arrow[d, "\sim"] \\
S^2 \arrow[ruu, "f_2"] \arrow[ru, "f_1"] \arrow[r, "f_0"'] & * = K(0,0)
\end{tikzcd}$$

In this problem session, we seek to construct a suitable candidate for $P_3(S^2)$. Now, by the definitions above, we know that it needs to satisfy

$$\pi_{\geq 4}(P_3(S^2)) = 0, \pi_3(P_3(S^2)) = \pi_3(S^2) = \mathbb{Z}, \pi_2(P_3(S^2)) = \mathbb{Z}, \pi_1(P_3(S^2)) = \pi_0(P_3(S^2)) = 0.$$

Furthermore, $F_3 = \mathrm{fib}(P_3(S^2) \to P_2(S^2))$ is a $K(\mathbb{Z}, 3)$. This means we have a fibration sequence

$$K(\mathbb{Z}, 3) \to P_3(S^2) \to P_2(S^2) = K(\mathbb{Z}, 2),$$

and we seek to figure out what $P_3(S^2)$ is.

Remark 1.8. Note that $K(\mathbb{Z}, 2), K(\mathbb{Z}, 3)$ are both topological abelian groups, so one might suspect we are solving a group extension problem here. There is no reason that $P_3(S^2)$ has to be a "group extension" itself, but the analogy would be closer with spectra.

A first guess someone might have is "$P_3(S^2) = K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$, we will quickly see that this is not the case. Indeed, note that

$$[S^2, K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)] = [S^2, K(\mathbb{Z}, 2)] \times [S^2, K(\mathbb{Z}, 3)],$$

and any map $S^2 \to K(\mathbb{Z}, 3)$ is null-homotopic by cellular approximation, in particular any map $S^2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$ cannot be an isomorphism on π_3, violating the condition above.

Now recall for these Postnikov towers, there is a **k-invariant map** that classifies the extension $P_2(S^3)$. (Here it is actually pretty relevant to note that we have these maps because we have principal fibrations, as $\pi_1(S^2) = 0$)

$$\kappa : P_2(S^2) = K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 4) = BK(\mathbb{Z}, 3).$$

Recall that $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$, and we have $[\mathbb{C}P^\infty, K(\mathbb{Z}, 4)] = H^4(\mathbb{C}P^\infty; \mathbb{Z})) = \mathbb{Z}$. Here κ is sent to 0 if and only if $P_2(S^3) \simeq K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 3)$, which we ruled out above.

Question 1.9. Which **non-zero number** k does κ represent in $\mathbb{Z} = H^4(\mathbb{C}P^\infty; \mathbb{Z})$?

One way to tackle this as follows - since $f_3 : S^2 \to P_3(S^3)$ is a $\pi_{\bullet \leq 3}$-equivalence, the Hurewicz theorem (or obstruction theory) implies that $H^3(P_3(S^2)) = H^3(S^2) = 0$.

Now, since $k \neq 0$, we have an induced diagram given by

$$\begin{tikzcd}
K(\mathbb{Z},3) \arrow[r] \arrow[d] & P_3 \arrow[r] \arrow[d] & K(\mathbb{Z},2) \arrow[r, "\kappa"] \arrow[d, "="] & K(\mathbb{Z},4) \arrow[d] \\
K(\mathbb{Z}/k,3) \arrow[r] & \tilde{P}_3 \arrow[r] & K(\mathbb{Z},2) \arrow[r] & K(\mathbb{Z}/k,4) = BK(\mathbb{Z}/k;3)
\end{tikzcd}$$

It follows that the induced $K(\mathbb{Z},2) \to K(\mathbb{Z}/k,4)$ is null-homotopic (it is k mod k, so represents 0 in $H^4(\mathbb{C}P^\infty;\mathbb{Z}/k) = \mathbb{Z}/k$). It then follows that

$$\tilde{P}_3 = K(\mathbb{Z};2) \times K(\mathbb{Z}/k;3).$$

Suppose for contradiction k is not a generator, then $K(\mathbb{Z}/k,3)$ is not contractible. Hence the map $P_3 \to \tilde{P}_3 \xrightarrow{\text{project}} K(\mathbb{Z}/k,3)$ is not null-homotopic and represents non-zero element in $H^3(P_3;\mathbb{Z}/k)$. On the other hand, $H^3(P_3;\mathbb{Z}/k) \cong H^3(S^2;\mathbb{Z}/k) = 0$, reaching a contradiction.

This shows that κ represents a generator of $H^4(\mathbb{C}P^\infty;\mathbb{Z})$.

2 Week 2: Homotopical Categories, Homotopy Limits and Colimits

2.1 Lecture January 27th, 2025 by Zixu Wang, Expanded by Mattie Ji

In this lecture, we will:

1. First discuss **homotopical categories**.
2. We then discuss the associated **homotopy category** of a **homotopical category**.
3. We will then recall the concept of **Kan extensions** to discuss **derived functors**.
4. We will then use derived functors to discuss **deformations** and finally to **homotopy (co)limits**.

The contents of this section has been expanded after the initial lecture.

2.1.1 Homotopical and Homotopy Categories

Definition 2.1. A homotopical (or model) category is a category M equipped with a wide subcategory (meaning it contains all the objects of M) W such that if

$$h \circ g, g \circ f \in W \implies f, g, h, h \circ g \circ f \in W.$$

This is called the "2-out-of-6" property. Pictorially, we think of this as:

$$\begin{tikzcd} \bullet \arrow[dr, "g\circ f"'] & \bullet \arrow[dr, "h\circ g"] & \\ & \bullet & \bullet \end{tikzcd} \implies \begin{tikzcd} \bullet \arrow[r, "f"] \arrow[dr, "g\circ f"'] \arrow[drr, "h\circ g\circ f"] & \bullet \arrow[d, "g"] \arrow[dr, "h\circ g"] & \\ & \bullet \arrow[r, "h"'] & \bullet \end{tikzcd}.$$

W is called the **weak equivalences**.

Remark 2.2. Those familiar with model categories may know the "2-out-of-3" properties, which is - if any of $f, g, g \circ f$ are in W, so is the third. It turns out the 2-out-of-6 properties implies the 2-out-of-3 properties and is stronger. A model category has more axioms, and it turns out they collectively imply a model category is homotopical.

The following is an example of a homotopical category.

Example 2.3. Any category $\mathcal{C}$ can be regarded as a homotopical category if we take the weak equivalences to be isomorphisms. We call this a **minimal homotopical category**. For this to be a valid homotopical category, we need to check that

$$h \circ g, g \circ f \text{ are isomorphisms} \implies f, g, h, h \circ g \circ f \text{ are isomorphisms.}$$

Here we do one example and leave the rest as exercises, observe that

$$((h \circ g)^{-1} \circ h) \circ g = id \text{ and } g \circ (f \circ (gf)^{-1}) = id,$$

which shows g is an isomorphism, and so on.

Here is another example closer to topology.

Example 2.4. Let $\mathcal{C} = \mathrm{Top}$ be the category of topological spaces. We can regard Top as a homotopical categoory if we take W to be homotopy equivalences. We can directly check that homotopy equivalences have 2-out-of-6 properties. Alternatively, we can consider a new category hTop where morphisms are homotopy equivalences of maps, hTop is a minimal homotopical category by isomorphisms (using Example 2.3. The natural map $\mathrm{Top} \to \mathrm{hTop}$ implies that Top is also homotopical with the constraints above.

Some more examples include (Top, weak homotopy equivalences) and (Top_*, based weak homotopy equivalences).

Here is an example that is closer to algebra.

Definition 2.5. A **quasi-isomorphism** is a chain map $f : C \to D$ (between chain complexes) that induces an isomorphism $f_* : H_n(C) \to H_n(D)$ for all n.

Example 2.6. Let $\mathrm{Ch}(R)$ be the category of unbounded chain complexes of R-modules, and W be quasi-isomorphisms, then they form a homotopical category. Quasi-isomorphisms should be think of as the algebraic analogue of weak homotopy equivalences.

Now that we have talked about homotopical categories, we can talk about **homotopy categories**.

Definition 2.7. Let $(\mathcal{C}, W)$ be a homotopical category, the **homotopy category** $\mathrm{Ho}(\mathcal{C})$ is the formal localization of $\mathcal{C}$ at the subcategory W. That is:

- Objects in $\mathrm{Ho}\,\mathcal{C}$ are the same as $\mathcal{C}$.
- Morphisms in $\mathrm{Ho}\,\mathcal{C}$ includes morphisms in $\mathcal{C}$ and formally adjoining inverses to all morphisms in W, and closing under composition.

We also have a natural localization functor $\gamma : \mathcal{C} \to \mathrm{Ho}\,\mathcal{C}$. This data can be equivalently defined as a universal property: for any functor $F : \mathcal{C} \to \mathcal{D}$ that sends weak equivalences to isomorphisms, F can be uniquely factored to γ:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\gamma"] \arrow[dr, "F"'] & \mathrm{Ho}(\mathcal{C}) \arrow[d, dashed, "\exists! G"] \\ & \mathcal{D} \end{tikzcd}$$

We say $(\mathcal{C}, W)$ is **saturated** if all morphisms that become isomorphisms in the localization are weak equivalences originally.

Although we worked with homotopical categories as opposed to model categories, it is true that all model categories are saturated.

Example 2.8. If W is the class of weak homotopy equivalences in Top_*, then $\mathrm{Ho}(\mathrm{Top}_*)$ is called the **homotopy category of spaces** or the **homotopy category** for short.

Here is another example that connects to topology.

Definition 2.9. If W is the class of quasi-isomorphisms in $\mathrm{Ch}(R)$, then $D(R) := \mathrm{Ho}(\mathrm{Ch}(R))$ is called the **derived category** of R.

We also have a notion of maps between homotopical categories.

Definition 2.10. We say a functor between two homotopical categories is **homotopical** if it preserves weak equivalences. By the universal property of localization above, this also induces a map between their homotopy categories. Pictorially depicted as:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d] \arrow[dr] & N \arrow[d] \\ \mathrm{Ho}(M) \arrow[r, dashed, "F'"'] & \mathrm{Ho}(N) \end{tikzcd}$$

Example 2.11. The functor $\mathrm{Top}^{\text{Path-Connected}} \to \mathrm{Grp}$ given by $X \to \pi_n(X)$ sends weak homotopy equivalences to isomorphisms, and is hence a homotopical functor. The functor $\mathrm{Ch}(R) \to \mathrm{Grp}$, $A \mapsto H_n(A)$ also similar is a homotopical functor (where $\mathrm{Ch}(R)$ uses quasi-isomorphisms and Grp uses isomorphisms).

Here following is a non-example.

Example 2.12. Limits and colimits are in general not homotopical. For a functor $F : \mathcal{C}^I \to \mathcal{C}$, it is not true this preserves weak equivalences in general. For example, consider the following two pushouts

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd} \qquad \begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & * \end{tikzcd}$$

We can see before taking the pushouts, the 3 terms are all weakly homotopy equivalent, but their pushouts are not.

2.1.2 Homotopical Approximations with Derived Functors

Given that there are functors that are not homotopical, we would like to try to approximate them in some sense.

Definition 2.13. Given functors $F : \mathcal{C} \to E$, $K : \mathcal{C} \to \mathcal{D}$. A **left Kan extension of F along K** is a functor $\mathrm{Lan}_K F : \mathcal{D} \to E$ with a natural transformation $\eta : F \implies \mathrm{Lan}_K F \circ K$ such that for any other such pair $(G : \mathcal{D} \to E, \gamma : F \implies G \circ K)$, the natural transformation γ factors uniquely through η.

Pictorially, suppose we have a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & \arrow[d, phantom, "\Downarrow \gamma"] & E \\ & \mathcal{D} \arrow[ur, "D"'] & \end{tikzcd}$$

Then it factors uniquely as a diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] \arrow[rr, phantom, shift right=6, "\Downarrow_\gamma"] & & E \\
& \mathcal{D} \arrow[ur, "\mathrm{Lan}_K F" description] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, shift right=4, "\Longrightarrow"] &
\end{tikzcd}$$

Dually we can also define **right Kan extension** of F along K, which is a functor $\mathrm{Ran}_K F : D \to E$ with a natural transformation $\epsilon : \mathrm{Ran}_K F \circ K \implies F$ such that any other such pairs $(G : \mathcal{D} \to E, \delta : G \circ K \implies F)$, δ factors uniquely through ϵ. (Notice here the natural transformation goes in another direction).

Example 2.14 (Example 1.4.1 of [Rie14])**.** Colimit is a left Kan extension! Indeed, let $\mathbb{1}$ denote the terminal category (i.e. 1 object, 1 morphism). For any category $\mathcal{C}$, there is an unique functor $K : \mathcal{C} \to \mathbb{1}$. Now let $F : \mathcal{C} \to E$ be another functor and consider the diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & E \\
& \mathbb{1} &
\end{tikzcd}$$

A left Kan extension is a functor $\mathrm{Lan}_K F : \mathbb{1} \to E$ with $\eta : F \to \mathrm{Lan}_K F \circ K$ satisfying the aforementioned universal property. Observe that any functor $G : \mathbb{1} \to E$ is exactly the data of specifying an object e in E. A natural transformation $\gamma : F \to G \circ K$ can be interpreted as follows - for any $f : c \to c'$ in $\mathcal{C}$, we have commutative diagrams:

$$\begin{tikzcd}
F(c) \arrow[r, "\gamma_c"] \arrow[d, "F(f)"'] & G \circ K(c) = e \arrow[d, "G \circ K(f) = \mathrm{id}_e"'] \\
F(c') \arrow[r, "\gamma_{c'}"'] & G \circ K(c') = e
\end{tikzcd},$$

which we see is equivalent to the data of a cocone. The universal property of η in the left Kan extension is the same as saying it is the universal cocone, in other words:

$$\mathrm{Lan}_K F, \eta \text{ is exactly the data of the colimit of } F : \mathcal{C} \to E.$$

Dually, the right Kan extension is the limit.

Remark 2.15. Riehl likes to say: "All concepts are Kan extensions", which is elaborated more in Section 1.4 of [Rie14].

Definition 2.16. Building on the set-up above, we say a functor $L : E \to E'$ **preserves a left Kan extension** $(\mathrm{Lan}_k F, \eta)$ if the **whiskered composite** $(L \circ \mathrm{Lan}_k F, L \circ \eta)$ is the left Kan extension of $L \circ F$ along K.

Pictorially:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[dr, "K"'] & E \arrow[r, "L"] & E' \\ & \mathcal{D} \arrow[u, "\mathrm{Lan}_K F"'] \arrow[ur, dashed, bend right] & \end{tikzcd} \quad \text{gives } L \circ \mathrm{Lan}_K F \simeq \mathrm{Lan}_K(L \circ F).$$

We can dually also define how to make right Kan extensions.

Definition 2.17. An **absolute Kan extension** is a Kan extension that is preserved by any functor.

Note that colimit is typically not an absolute Kan extension, because applying an additional functor need not preserve colimits in general.

We can now discuss how to approximate homotopical functors using what are called **derived functors**.

Definition 2.18. A **total left derived functor** $\mathbf{L}F$ of a functor F between homotopical categories M and N is a **right Kan extension** $\mathrm{Ran}_\gamma \, \delta \circ F$ in the context:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[dr, "\gamma"'] & N \arrow[r, "\gamma"] & \mathrm{Ho}\, N \\ & \mathrm{Ho}\, M \arrow[u, dashed, phantom, "\Uparrow"] \arrow[ur, dashed, "\mathrm{Ran}_\gamma(\gamma \circ F)"'] & \end{tikzcd}$$

Here γ and δ are the respective localization maps.

Dually, we can also define the total right derived functor $\mathbf{R}F := \mathrm{Lan}_\gamma \, \delta \circ F$ (Note here we switch left and right in the definition).

Remark 2.19. By the universal property of localizations, such a left derived functor is equivalent to a homotopical functor $\mathbb{L}F : M \to \mathrm{Ho}\, N$ (via the uniqueness and existence of these liftings). If we can also lift the codomain to N, we call these lifts **left derived functors**.

More formally describing the remark above:

Definition 2.20. Let $F : M \to N$ be a functor between homotopical categories. A **left derived functor** of F is a homotopical functor $\mathbb{L}F : M \to N$ with a natural transformation $\lambda : \mathbb{L}F \implies F$ such that the natural transformation $\delta\lambda : \delta \circ \mathbb{L}F \implies \delta \circ F$ (obtained by post-composing δ) and the induced functor $\mathrm{Ho}\, M \to \mathrm{Ho}\, N$, from universal property of localizations, form a **total left derived functor** of F.

Here are some pictures to illustrate some of the discussions above.

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[dr, "LF"'] & N \arrow[d, "\gamma"] \\ & \mathrm{Ho}(N) \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, dashed, "\mathbb{L}F"] \arrow[dr, "LF"'] & N \arrow[d, "\gamma"] \\ & \mathrm{Ho}(N) \end{tikzcd}$$

2.1.3 Deformation and Homotopy (Co)Limits

With all the discussions in the previous section, one might wonder - what are some examples where these derived functors actually show up in the topological world. We will see how they can help to construct homotopy (co)limits in this part.

First we make the following observation pointed out in [Shu09].

Definition 2.21 (See Section 2 of [Shu09]). Let M be a homotopical category and $\mathcal{C}$ be a small category. The functor category $\mathrm{Fun}(\mathcal{C}, M)$ has an induced structure as a homotopical category where a natural transformation $\alpha : F \to G$ is **weak equivalence** if for each $c \in \mathcal{C}$, the map $\alpha_c : F(c) \to F(c)$ is a weak equivalence in M.

A natural transformation that is a weak equivalence in thsi case is called a **natural weak equivalence**.

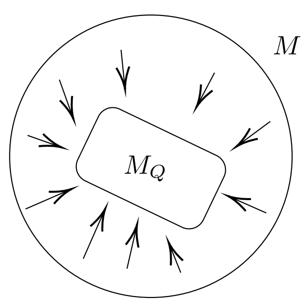
From here we discuss a "deofrmation" of a homotopical category.

Definition 2.22. A **left deformation** on a homotopical category M is an endofunctor $Q : M \to M$ together with a natural weak equivalence $q : Q \implies \mathrm{id}$. Note that any such functor is homotopical.

Definition 2.23. Let Q be a left deformation on M and M_Q denotes any full subcategory of M containing the image of Q. Here M_Q is called the **associated subcategory of cofibrant objects**.

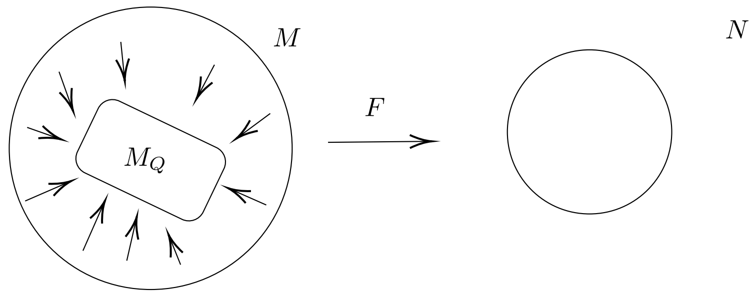
Proposition 2.24. The inclusion functor $M_Q \to M$ and the left deformation $Q : M \to M_Q$ induces an equivalence between $\mathrm{Ho}(M)$ and $\mathrm{Ho}(M_Q)$.

Pictorially, the proposition above can be thought of as saying there is a "deformation retract":



Definition 2.25. Let M and N be homotopical categories. A left deformation for a functor $F : M \to N$ between homotopical categories consists of a left deformation for M such that F is homotopical on an **associated subcategory of cofibrant objects**.

Here is a pictorial interpretation of the definition above:



Now we connect back to the previous section by showing that said deformations can construct derived functors.

Theorem 2.26 (Theorem 2.2.8 of [Rie14])**.** Suppose $F : M \to N$ has a left deformation $q : Q \implies \mathrm{id}$, then $\mathbb{L}F := F \circ Q$ is a left derived functor of F.

We can also dually define **right deformations** which has an analogous theorem as well.

Definition 2.27. A **right deformation** on a homotopical category M is an endofunctor $R : M \to M$ together with a natural weak equivalence $q : \mathrm{id} \implies R$. Note that any such functor is homotopical. Let M_R denote any full subcategory of M containing the image of R, thnen this is called the **associated subcategory of fibrant objects**. For two homotopical categories M, N, a right deformation of $F : M \to N$ is a right deformation R on M such that $F|_{M_R}$ is homotopical.

Let $F : M \to N$ be a functor between homotopical categories. A **right derived functor** of F is a homotopical functor $\mathbb{R}F : M \to N$ with a natural transformation $\eta : F \implies \mathbb{R}F$ such that the natural transformation $\eta\delta_M : F \circ \delta_M \implies \mathbb{R}F \circ \delta_M$ (defined by precomposing with localization $\delta_M : M \to \mathrm{Ho}(M)$) and the induced functor $\mathrm{Ho}(M) \to \mathrm{Ho}(N)$ forms a **total right derived functor** of F.

Remark 2.28. In other languages, left and right Quillen functors are respectively examples of left derived and right derived functors.

Theorem 2.29. Suppose $F : M \to N$ has a right deformation $q : \mathrm{id} \implies R$, then $\mathbb{R}F := F \circ R$ is a right derived functor of F.

Theorem 2.30 (Theorem 2.2.11 of [Rie14])**.** We say a functor F is left (resp. right) deformable if it is a left (resp. right) deformation. Suppose there is a pair of adjunctions $F : M \to N$ and $G : N \to M$ such that F is left deformable and G is right defomrable. Then there is an adjunction:

$$\mathbb{L}F : \mathrm{Ho}\, M \rightleftarrows \mathrm{Ho}\, N : \mathbb{R}G$$

that is "unique" with a certain compatibility condition with localizations.

Example 2.31. CW approximation is a left deformation on the homotopical category of compact generated weakly Hausdorff (CGWH) spaces with weak homotopy equivalences. To make the CW approximation functorial, we usually take the candidate to be $|\operatorname{Sing}(\bullet)|$.

2.1.4 Constructing Homotopy (Co)limits

In this part, we will be constructing homotopy (co)limits. Before we do so, we should first explain some motivation behind homotopy (co)limits. Recall from Example 2.12 that (co)limits in general do not preserve weak equivalences, which means they do not really "respect homotopical structures". The theory we set up earlier suggests a way to "approximate (co)limits"by functors that are actually homotopical. Let $\mathcal{D}$ be a small category and M a homotopical category. Recall that we can construct (co)limits for any functor $\mathcal{D} \to M$.

Definition 2.32. Let $\mathrm{colim}, \lim : M^{\mathcal{D}} \to M$. We define:

1. **homotopy colimit** $\mathrm{hocolim} : M^{\mathcal{D}} \to M$ as the left derived functor of colim.
2. **homotopy limit** $\mathrm{holim} : M^{\mathcal{D}} \to M$ as the right derived functor of $\lim$.

One might ask if the homotopy (co)limits in the definition above as well-defined and unique. This is a good question, we first note that:

Remark 2.33. If M is saturated, then the homotopy (co)limits, if exist, will have the same homotopy type.

Showing that they exist is more subtle. To do this, we will need Theorem 2.26, which tells us we can construct **left derived functors** (and dually right derived functors) as long as there are left (respectively right) deformations. We will define these deformations on a subcollection of homotopical categories which has the desired conditions. These are known as **simplicial model category**.

For our purposes, one can keep the following important examples in mind instead of the technical definition of a simplicial model category:

1. The category of simplicial sets sSet and pointed simplicial sets sSet_*.
2. The category CGWH of compactly generated weakly Hausdorff spaces.

The approach taken in Riehl's book is that Riehl first specified what formal properties all simplicial model categories had, and developed the deformations using consequences of these formal properties, and gave the precise definition later. For our purposes, we include the notion of a **simplicial model category** for completeness:

Definition 2.34 (See II.2 of [Qui67], Chapter 3 of [Rie14], and 4.2 of [GS07]). Recall the category of simplicial sets, denoted sSet, is symmetric monoidal with Cartesian product $\times$, with symmetric monoidal unit.

1. A **simplicial category** $\mathcal{C}$ is a category enriched over sSet. We define this more generally, let $(\mathcal{V}, \times, 1)$ be a symmetric monoidal category whose operation is written as $\times$. A $\mathcal{V}$-category $\mathcal{C}$ (called enriched in $\mathcal{V}$) is composed of:

   (a) Objects in $\mathcal{C}$, denoted $\mathrm{obj}(\mathcal{C})$.

   (b) For each pair of objects $x, y \in \mathcal{C}$, an object $\mathrm{Map}_{\mathcal{C}}(x,y) \in \mathcal{V}$.

   (c) A morphism $1 \to \mathcal{M}\dashv_{\sqrt{\mathcal{C}}}(x,x)$, thought of as the identity map, for each $x \in \mathrm{obj}(\mathcal{C})$.

   (d) For each x, y, z, there is a morphism $\circ : \mathrm{Map}_{\mathcal{C}}(y,z) \times \mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{C}}(x,z)$.

   such that the chosen collection fulfills the obvious axioms that the composition is associative and composing with identity gives the same map.

2. Note that sSet is actually enriched over itself, but $\mathrm{Map}_{sSet}(x,y)$ is actually not the same as $\mathrm{Hom}_{sSet}(x,y)$. In general, if for all $v \in \mathcal{V}$, the functor $\bullet \times v : \mathcal{V} \to \mathcal{V}$ has a right-adjoint, we can parametrize the right adjoints to define an internal hom $\mathrm{Map}_{\mathcal{V}}(-,-) : \mathcal{V}^{op} \times \mathcal{V} \to \mathcal{V}$, so $\mathcal{V}$ is enriched over itself. Such categories are called **closed symmetric monoidal**.

3. Recall that sSet is symmetric monoidal with Cartesian product $\times$, has a terminal object, furthermore for each $K \in \mathrm{sSet}$, morphisms of simplicial sets have the natural structure of being a simplicial set as well. Let $\mathcal{V}$ be any such category (which are called **closed symmetric monoidal categories** more generally[a]).

   We say a $\mathcal{V}$-category $\mathcal{C}$ is tensored over $\mathcal{V}$ if for each $v \in \mathcal{V}$ and $m \in \mathcal{C}$, there is an object called $v \otimes m \in \mathcal{C}$ with a compatible isomorphism
   $$\mathrm{Map}_{\mathcal{C}}(v \otimes m, n) \cong \mathrm{Map}_{\mathcal{V}}(v, \mathrm{Map}_{\mathcal{C}}(m, n))$$
   for all $n \in \mathcal{C}$.

   We say a $\mathcal{V}$-category $\mathcal{C}$ is cotensored over $\mathcal{V}$ if for each $v \in \mathcal{V}$ and $n \in \mathcal{C}$ there is an object $n^v \in \mathcal{M}$ with a compatible isomorphism
   $$\mathrm{Map}_{\mathcal{C}}(m, n^v) \cong \mathrm{Map}_{\mathcal{V}}(v, \mathrm{Map}_{\mathcal{C}}(m, n)).$$
   for all $m \in \mathcal{C}$.

   In this sense, "being tensored/cotensored" is really akin to saying there is an analog of "2-variable adjunction for tensor and dual".

4. A **simplicial model category** is a model category M that is simplicial, tensored and cotensored over sSet, such that the following axiom known as **Quillen's SM7** holds:

   For $j : A \to B$ a cofibration and $q : X \to Y$ a fibration. They induce natural maps
   $$i_* : \mathrm{Map}_M(B, Y) \to \mathrm{Map}_M(A, Y) \text{ and } j_* : \mathrm{Map}_M(A, X) \to \mathrm{Map}_M(A, Y)$$
   Consider now the pullback diagram (note being a model category implies it has pullbacks)
   $$\begin{tikzcd} P \arrow[r] \arrow[d] & \mathrm{Map}_M(A, X) \arrow[d, "i_*"] \\ \mathrm{Map}_M(B, Y) \arrow[r, "j_*"'] & \mathrm{Map}_M(A, Y) \end{tikzcd}$$
   There are also natural maps $\mathrm{Map}_M(B, X) \to \mathrm{Map}_M(B, Y)$ and $\mathrm{Map}_M(B, X) \to \mathrm{Map}_M(A, X)$, which induces a map:
   $$\begin{tikzcd} \mathrm{Map}_M(B, X) \arrow[dr, "\phi"] \arrow[drr] \arrow[ddr] & & \\ & P \arrow[r] \arrow[d] & \mathrm{Map}_M(A, X) \arrow[d, "i_*"] \\ & \mathrm{Map}_M(B, Y) \arrow[r, "j_*"'] & \mathrm{Map}_M(A, Y) \end{tikzcd}$$
   We require ϕ to be a **fibration of simplicial sets**. Furthermore, this is a weak equivalence if j or q was a weak equivalence in $\mathcal{M}$.

[a] being a bit loose here on how exactly this generalize

Theorem 2.35 (Lemma 3.8.6 of [Rie14])**.** A simplicial model category M has the properties that:

1. M is cocomplete, cocomplete, and saturated.
2. The subcategories of cofibrant and fibrant objects are preserved by tensors and cotensors.
3. M has a left deformation Q to cofibrant objects (cofibrant replacement) and a right deformation R to fibrant objects (fibrant replacement).
4. Every simplicial homotopy equivalence is a weak equivalence.
5. $\mathrm{Map}_M(-,-)$ preserves weak equivalences between cofibrant objects in the first variable, if the second variable is fibrant.
6. $\mathrm{Map}_M(-,-)$ preserves weak equivalences between fibrant objects in the second variable, if the first variable is cofibrant.

Now we move back to the setting of homotopy (co)limits, which are given as left/right derived functors

$$\mathrm{colim} : M^{\mathcal{D}} \to M \text{ and } \lim : M^{\mathcal{D}} \to M.$$

There is an explicit description of deformations for them using what are called **bar** and **co-bar** constructions.

Definition 2.36 (4.2 of [Rie14])**.**

1. Recall M being tensored over $\mathcal{V}$ gives a bifunctor $-\otimes- : \mathcal{V}\times M \to M$ (via Yoneda lemma).

   Given two functors $G : \mathcal{D}^{op} \to \mathrm{sSet}$ and $F : \mathcal{D} \to M$. The two-sided **simplicial bar construction** is a simplicial object $B_\bullet(G, \mathcal{D}, F) \in sM$ such that

   $$B_n(G, \mathcal{D}, F) \coloneqq \coprod_{d_0\to d_1\to\ldots\to d_n \text{ in } \mathcal{D}} Gd_n \otimes Fd_0.$$

   The degeneracy maps $B_n(G, \mathcal{D}, F) \to B_{n+1}(G, \mathcal{D}, F)$ corresponding to sending the copy at $d_0 \to d_1 \ldots \to d_n$ to $d_0 \to \ldots \to d_i \xrightarrow{identity} d_i \to \ldots \to d_n$ in the coproduct, for each i.

   The inner face maps also corresponding to sending the copy at $d_0 \to \ldots \to d_n$ to $d_0 \to \ldots d_{i-1} \to d_{i+1} \to \ldots d_n$, where the map $d_{i-1} \to d_{i+1}$ is the composition.

   For the n-th face map, we define a map $Gd_n \otimes Fd_0 \to Gd_{n-1} \otimes Fd_0$ by applying $G(-) \otimes Fd_0$ at the morphsim $d_n \to d_{n-1}$.

   For the 0-th face map, we define a map $Gd_n \otimes Fd_0 \to Gd_n \otimes Fd_1$ by applying the functor $Gd_n \otimes F(-)$ at the morphism $d_0 \to d_1$.

2. For any category $\mathcal{C}$ tensored over sSet, there is a **geometric realization** functor $|\bullet| : s\mathcal{C} \to \mathcal{C}$ given as follows. For a simplicial object $X_\bullet \in s\mathcal{C}$, one defines

   $$|X| \coloneqq \mathrm{coeq}\Big(\coprod_{f:[m]\to[n]\in\Delta^{op}} \Delta^n \otimes X_m \rightrightarrows^{f^*}_{f_*} \coprod_{[k]\in\Delta^{op}} \Delta^k \otimes X_k\Big).$$

   Here one map is given by $\Delta^n\otimes X_m \to \Delta^n\otimes X_n$ by applying the functor $\Delta^n\otimes X_\bullet$ along $[m] \to [n]$ in Δ^{op} (recall $X_\bullet : \Delta^{op} \to \mathcal{C}$), over all items in the coproduct. The other map is given by $\Delta^n\otimes X_m \to \Delta^m\otimes X_m$ by applying the functor $\Delta^\bullet \otimes X_m$ along $[n] \to [m] \in \Delta$ (recall $\Delta^\bullet : \Delta \to \mathrm{sSet}$ is the Yoneda embedding).

Remark 2.37. This really does recover the classical geometric realization.

3. We define the bar construction $B(G, \mathcal{D}, F)$ as the geometric realization of $B_\bullet(G, \mathcal{D}, F)$.

We can dually define **cosimplicial cobar construction**. For completeness, we outline them as follows.

Definition 2.38. 1. The **cosimplicial cobar construction** $C^\bullet(G, \mathcal{D}, F)$ is a cosimplicial object in M given by

$$C^n(G, \mathcal{D}, F) = \prod_{d_0 \to \ldots \to d_n \in \mathcal{D}} F d_n^{G d_0}$$

2. Given any cosimplicial object $X^\bullet$ of M, there is a **totalization functor** that sends $X^\bullet$ to $\operatorname{Tot} X^\bullet \in M$.

3. The **cobar construction** $C(G, \mathcal{D}, F)$ is the totalization of $C^\bullet(G, \mathcal{D}, F)$.

The big punchline of the above constructions is that they are actually appropriate deformations.

Theorem 2.39 (Theorem 5.1.1 of [Rie14])**.** Let M be a simplicial model category with cofibrant replacement $q : Q \implies 1$ and fibrant replacement $r : 1 \implies R$. Observe that universal properties define natural transformations

$$\epsilon_Q : B(\mathcal{D}, \mathcal{D}, Q-) \implies Q \text{ and } \eta_R : R \implies C(\mathcal{D}, \mathcal{D}, R-).$$

1. $B(\mathcal{D}, \mathcal{D}, Q-)$ with $q \circ \epsilon_Q$ is a left deformation of $\operatorname{colim} : M^{\mathcal{D}} \to M$.

2. $B(\mathcal{D}, \mathcal{D}, R-)$ with $\eta_R \circ r$ is a right deformation of $\lim : M^{\mathcal{D}} \to M$.

Theorem 2.26 and the above theorem implies that homotopy (co)limits may be described as follows:

Corollary 2.40. Let $F : \mathcal{D} \to M$ be a functor, then

$$\operatorname{hocolim}(F) \simeq B(*, \mathcal{D}, Q \circ F) \text{ and } \operatorname{holim}(F) \simeq C(*, \mathcal{D}, R \circ F).$$

2.1.5 Examples of Homotopy (Co)limits

The presentation above is quite categorical, but when we restrict to spaces the homotopy (co)limits have an equivalent and concrete formulation. This is explained quite well in Chapter 6 of [Rie14] and Chapter 1 of [Mal26].

For the remainder of this course, the following type of homotopy (co)limits of spaces are the main types we would use:

1. (Homotopy) Products and Coproduct.
2. Homotopy Pushouts and Pullbacks.
3. Homotopy Sequential Limits and Sequential Colimits.

Here we record constructions that will be helpful. The main point is that homotopy groups actually commute with the homotopy (co)limits (as opposed to the usual ones where this may fail).

Proposition 2.41. A homotopy (co)product of based spaces is the underlying (co)product of spaces.

Homotopy Pushouts and Pullbacks:

Definition 2.42. Given the diagram of spaces:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y \\ Z & \end{tikzcd}$$

The **homotopy pushout** is $Z \cup_X^h Y$ given by

$$Z \cup_X^h Y := Z \cup_{X \times \{0\}} (X \times [0,1]) \cup_{X \times \{1\}} Y$$

where the glueing is done by attaching $X \times \{0\}$ to Z along the map g and $X \times \{1\}$ to Y along the map f.

The **homotopy cofiber** of $f : X \to Y$ is the homotopy pushout of the special case when $Z = *$.

Remark 2.43. Strictly speaking, the homotopy pushout given above is the unbased version, but it does have the same homotopy type as the based homotopy pushout.

The homotopy pushout gives a Mayer-Vietoris result for homology as follows.

Proposition 2.44 (Lemma 1.5.6 of [Mal26])**.** There is a long exact sequence of the form

$$\ldots \to H_k(X) \to H_k(Y) \oplus H_k(Z) \to H_k(Z \cup_X^h Y) \to H_{k-1}(X) \to \ldots$$

Definition 2.45. Given a diagram of spaces:

$$\begin{tikzcd} & Y \arrow[d, "g"'] \\ X \arrow[r, "f"] & Z \end{tikzcd}$$

The **homotopy pullback** $X \times_Z^h Y$ is

$$X \times_Z^h Y := \{x \in X, y \in Y, \gamma : [0,1] \to Z, \gamma(0) = f(x), \gamma(1) = g(y)\}.$$

The **homotopy fiber** of $X \to Z$ is the special case when $Y = *$.

Proposition 2.46 (Lemma 1.5.19 of [Mal26])**.** The homotopy pullback is weakly equivalent to the strict pullback if one of $f : X \to Z$ or $g : Y \to Z$ is a **Serre fibration.**

Note from the definition above, we also get that:

Proposition 2.47. The homotopy pullback of maps $* \to X$ and $* \to X$ is the based loop space ΩX.

A nice property of the homotopy pullback is as follows.

Proposition 2.48 (Mayer-Vietoris of Homotopy Pullbacks, Lemma 1.5.22 of [Mal26])**.** There is a long exact sequence of homotopy groups

$$\ldots \to \pi_k(X \times_Z^h Y) \to \pi_k(X) \oplus \pi_k(Y) \to \pi_k(Z) \to \pi_{k-1}(X \times_Z^h Y) \to \ldots$$

Homotopy Sequential (Co)Limits

Definition 2.49. Consider a sequence of spaces

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \ldots$$

The **homotopy colimit** of this diagram is

$$\operatorname{hocolim}_i X_i \coloneqq \bigsqcup_{n \geq 0} (X_n \times [0,1])/((x_n, 1) \sim (f_n(x_n), 0).$$

For a sequence of spaces

$$\ldots \to X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

The **homotopy limit** of this diagram is

$$\operatorname{holim}_i X_i \coloneqq \{\gamma_n : [0,1] \to X_n, n \geq 0 \mid f_n(\gamma_n(1)) = \gamma_{n-1}(0)\}$$

Proposition 2.50 (Lemma 1.5.22 of [Mal26])**.** If each f_i is a Serre fibration, then $\operatorname{holim}_i X_i$ is weakly equivalent to $\lim_i X_i$.

We summarize the properties of homotopy sequential (co)limits below.

Theorem 2.51 (See 1.5 of [Mal26])**.**

1. Homotopy sequential colimits of spaces $\mathrm{hocolim_i X}$ commute with homology, homotopy groups, and more generally $[Y, -]$ for any finite CW complex Y.
2. For cohomology, there is a $\lim^1$ short exact sequence

$$0 \to \lim_{n\to\infty}{}^1 H^{k-1}(X_n) \to H^k(\operatorname{hocolim}_{n\geq 0} X_n) \to \lim_{n\to\infty} H^k(X_n) \to 0.$$

3. For homotopy sequential limits of spaces, there is a $\lim^1$ short exact sequence

$$0 \to \lim_{n\to\infty}{}^1 \pi_{k+1}(X_n) \to \pi_k(\operatorname{hocolim}_{n\geq 0} X_n) \to \lim_{n\to\infty} \pi_k(X_n) \to 0.$$

3 Week 3: Stable Phenomena in Spaces by Riley Shahar

Today we will be talking about **stable phenomena in spaces**. This is a motivational talk in some sense. There are some nice machineries we will build, but a lot of this is done on the point-set level which will motivate wh we are going into **spectra** later.

The outline of the talk is as follows:

1. We will first introduce a convenient category of spaces CGWH, emphasizing on their formal properties.
2. We will then discuss suspension and loop spaces.
3. We will then talk about the **Freudenthal Suspension Theorem**.
4. Finally, we will talk about the **Blakers-Massey Theorem**, which can be thought of as a form of **homotopy excision**.

The outline of this talk follows closely to Chapter 2 of [Sto22].

3.1 The CGWH Category

Before we talk about a nice property of spaces, we should ask:

Question 3.1. What is wrong with taking the category of all (pointed) topological spaces (Top and Top_*)?

There are some problems with these two categories, namely:

1. They fail to be Cartesian closed under the natural symmetric monoidal operation on them as smash products.
2. (Categorical) Products of CW complexes do not have a natural cell structure.
3. The smash product may not be associative!

One solution is to work with what are so-called "**compactly generated weak Hausdorff**" spaces. We will not define them, but instead we will tell you some formal properties they have:

1. CGWH is a full subcategory of Top.
2. CGWH is complete and cocomplete, and it is Cartesian closed (in fact it is Cartesian closed in an enriched sense).
3. **Warning:** The forgetful functor $\mathrm{CGWH} \to \mathrm{Top}$ does not need to preserve limits or colimits (in fact, even taking to Set may not resolve the issue).
4. The warning is nice, however, since product of CW complexes is a CW complex now.

Notation: From now on, we will write Top to mean CGWH.

We will now talk about some relevant constructions on these spaces.

Definition 3.2. Given $(X, x_0), (Y, y_0)$ pointed spaces, we define:

1. The **wedge sum** is
$$X \vee Y := X \sqcup Y / x_0 \sim y_0.$$
2. The **based function space** is
$$\mathcal{F}(X, Y) \text{ as a subspace } Y^X \text{ of based point preserving maps,}$$
pointed with the constant map. Note here CGWH (what we write now as Top) is Cartesian closed, which gives the data of a topology on Y^X.

3. The **smash product** is

$$X \wedge Y := X \times Y / X \vee Y.$$

Proposition 3.3. The category Top_*, by which we mean pointed CGWH spaces, is symmetric monoidal with $\wedge$ as the symmetric monoidal product, $\mathcal{F}$ as the inner hom, and in fact this holds in an enriched sense that

$$\mathcal{F}(X \wedge Y, Z) \cong \mathcal{F}(X, \mathcal{F}(Y, Z))$$

as pointed spaces.

Observe the interval $[0, 1]$ is clearly in CGWH.

Definition 3.4. From here, we define

$$[X, Y]_* = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Top}_*)}(X, Y)$$

for the set of based homotopy classes of maps.

Proposition 3.5. $\pi_0 \mathcal{F}(X, Y) = [X, Y]_*$.

Proof. Observe that by adjunction,

$$\begin{aligned} \pi_0 \mathcal{F}(X, Y) &= [S^0, \mathcal{F}(X, Y)]_* \\ &\cong [S^0 \wedge X, Y]_* \\ &\cong [X, Y]. \end{aligned}$$

■

3.2 Suspension and Loop Spaces

Now we will talk about suspension and loop spaces.

Definition 3.6. Let X be a pointed space.

- The **(reduced) suspension** of X, denoted ΣX, is the smash product $X \wedge S^1$.
- The **loop space** of X, denoted ΩX, is the mapping space $\mathcal{F}(S^1, X)$.

By the adjunction outlined previously, we have that.

Proposition 3.7. $\Sigma(-)$ is left adjoint to $\Omega(-)$. In other words, we have that

$$\mathcal{F}(\Sigma X, Y) \cong \mathcal{F}(X, \Omega Y).$$

The (unreduced) suspension SX can be thought of as the pushout of two maps $X \to CX$, where CX denotes the cone of X, so we have a pushout diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & SX \end{tikzcd}$$

Since CX is contractible, we may wish to think of ΣX as a "homotopy version" of this. Indeed, we have a **homotopy pushout**

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

Similarly, we can describe ΩX as the pullback

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX \arrow[d, "\text{end}"'] \\ PX \arrow[r, "\text{start}"] & X \end{tikzcd}$$

where $PX := F(I, X)$. This realizes ΩX as the **homotopy pullback** in

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd}$$

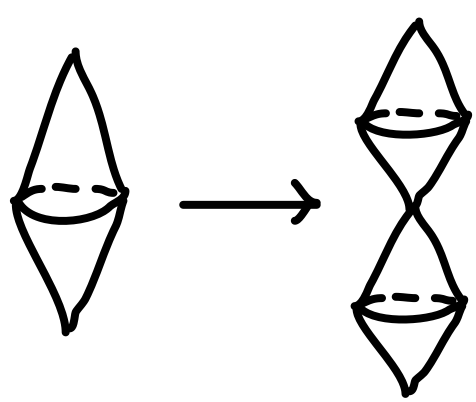
We also make another observation.

Proposition 3.8. From the proof that the fundamental group is a group, we can see that ΩX is a "group up to homotopy", i.e. it is an H-group.

Dually we observe there is a natural map by **pinching in the middle**

$$\Sigma X \to \Sigma X \vee \Sigma X.$$

This realizes ΣX as an **H-cogroup**. Here is a picture



By **Yoneda's Lemma**, we have the following.

Proposition 3.9. A space X is an H-group if and only if $[-, X]_*$ factors through Grp. X is an H-cogroup if and only if $[X, -]$ factors through Grp. So we have natural group structures on

$$[X, \Omega Y]_* \cong_{\mathrm{Grp}} [\Sigma X, Y]_*.$$

3.3 The Freudenthal Suspension Theorem

Recall for reduced cohomology, we have a natural isomorphism

$$\tilde{H}^k(X) \to \tilde{H}^{k+1}(\Sigma X).$$

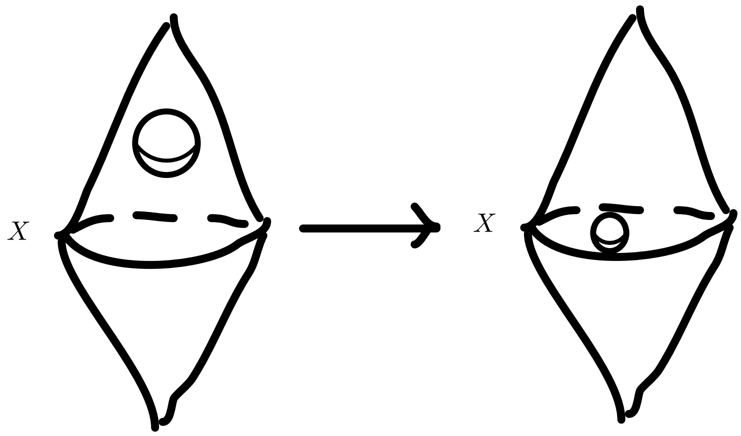
However, this **Cannot Hold for homotopy**! But we do still have an induced map

$$\pi_k(X) \to \pi_{k+1}(\Sigma X)$$

given by applying the suspension functor

$$\Sigma : [S^k, X] \to [S^{k+1}, \Sigma X] = [S^k, \Sigma X].$$

However, one can imagine the following **thought experiment** on what happens when k is sufficiently large.



Where if X is sufficiently connected, we may be able to move a high dimensional sphere down from ΣX to the "center", the part belong to the copy of X, which yields an isomorphism of sorts. We make them precise with the following discussion.

Definition 3.10. A space X is **n-connected** if $\pi_{\leq n}(X) = 0$. A map $f : X \to Y$ is **n-connected** if it induces an isomorphism on $\pi_{<n}$ and a surjection on π_n.

Theorem 3.11 (Freudenthal Suspension Theorem)**.** Let X be n-connected, then the canonical map

$$\pi_k(X) \to \pi_{k+1}(\Sigma X)$$

is an isomorphism for $k < 2n + 1$, and a surjection when $k = 2n + 1$.

As a corollary we see that:

Corollary 3.12. If X is n-connected, then ΣX is $(n + 1)$-connected.

We also see that:

Corollary 3.13. The sequence

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \ldots \to \pi_{k+j}(\Sigma^j X) \to \ldots$$

is eventually constant.

From the previous corollary, we can define the following.

Definition 3.14. The **stable value** of the sequence above is called **the k-th stable homotopy group** of X. We write this as $\pi_k^{st}(X)$.

We also obtain the following statement about homotopy classes of maps:

Corollary 3.15. If X is a finite-dimensional CW complex, then sequence

$$[X, Y]_* \to [\Sigma X, \Sigma Y]_* \to \ldots$$

eventually stabilizers.

Definition 3.16. We write $\{X, Y\}$ for the stable value of the sequence above. Note this is an abelian group (essentially by Proposition 3.9 and the virtue of Eckmann-Hilton)!

Remark 3.17. There is a space we can make as

$$QX := \text{colim}(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \ldots)$$

The homotopy groups of QX are the same as the stable homotopy groups of X.

3.4 The Blakers-Massey Theorem

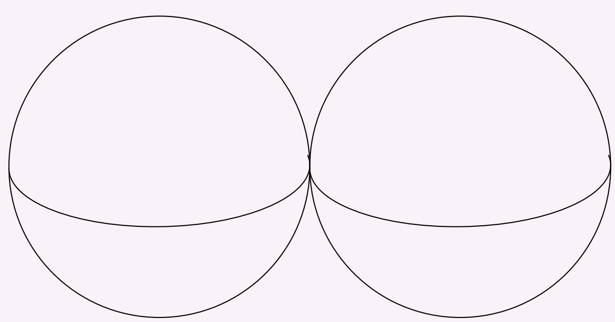
Recall that cohomology satisfies **excision**. Recall this means that if $A, B \subseteq X$ with $int(A) \cup int(B) = X$, then the inclusion

$$(A, A \cap B) \to (X, B)$$

induces an isomorphism on cohomology.

Excision fails for unstable homotopy theory, however.

Example 3.18. Write $X = S^2 \wedge S^2$.



Let A be the top hemispheres (with a little bit of overlap), and B be the bottom hemispheres (with a little bit of overlap). Then $A \cap B \simeq S^1 \wedge S^1$. From here we see the inclusion

$$(A, S^1 \vee S^1) \hookrightarrow (X, B)$$

has that $\pi_2(X, B) \cong \pi_2(S^2 \vee S^2)$ is abelian, but $\pi_2(A, S^1 \vee S^1) \cong \pi_1(S^1 \vee S^1) \cong F_2$ is the free group on 2 generators.

We would like stable homotopy groups to satisfy a version of **excision** though. Observe that Freudenthal's suspension theorem is really a statement about the connectivity of the map $X \to \Omega\Sigma X$.

We can suitably consider a generalization of this. Given maps $f : Y \to A$ and $g : Y \to B$. Let X denote the homotopy pushout of f and g:

$$\begin{tikzcd}
Y \arrow[r, "g"] \arrow[d, "f"] & B \arrow[d, "h_B"'] \\
A \arrow[r, "h_A"] & X
\end{tikzcd}$$

Let Z be the pullback of h_A and h_B, Observe there is now a natural map

$$\begin{tikzcd}
Y \arrow[dr, dashed, "\phi"] \arrow[drr, "g"] \arrow[ddr, "f"'] & & \\
& Z \arrow[r] \arrow[d] & B \arrow[d, "h_B"'] \\
& A \arrow[r, "h_A"] & X
\end{tikzcd}$$

Theorem 3.19 (Blakers-Massey)**.** If f is m-connected and g is n-connected, then the map $\phi : Y \to Z$ is $(m + n + 1)$-connected.

This proves Freudenthal's Suspension Theorem if we take $A = B = *$!

Here is another corollary, which is the question of how close the homotopy fiber is equivalent to an actual fiber.

Corollary 3.20. Suppose $g : X \to Y$ is n-connected with X being m-connected. Take the homotopy cofiber

$$X \xrightarrow{g} Y \xrightarrow{c} Cg.$$

We can also take the homotopy fiber of c, which fits in

$$F \to Y \to Cg.$$

The connectivity of the natural map $X \to F$ is $(m + n)$-connected.

Remark 3.21. This will imply that as we stabilizer, cofiber and fiber sequences should agree!

4 Week 4: Generalized Cohomologies by Angel Yuan

The outline of today's talk is as follows:

1. We will first talk about **Generalized Cohomology Theories**.
2. We will then discuss **K-theory** as an example of generalized cohomology theories.
3. We will then talk about **infinite loop spaces** and Ω-spectra.
4. Finally we will talk about how **Brown representability** connects them.

4.1 Generalized Cohomology Theories

Definition 4.1. A (reduced) **generalized cohomology theory** is a sequence of contravariant functors h^n : $\mathrm{Top}_*^{op} \to \mathrm{Ab}$ satisfying axioms:

1. (Homotopy Invariance) If $f \sim g$ are based homotopic, then $h^n(f) = h^n(g)$. (ie. the functor passes to hTop_*).
2. (Exactness) If $A \to X \to Q$ is a cofiber, then there is an exact sequence
$$h^n(Q) \to h^n(X) \to h^n(A).$$
3. (Suspension Isomorphisms) There is a natural isomorphism
$$h^{n+1}(\Sigma -) \xrightarrow{\cong} h^n(-).$$
4. (Wedge Axioms) The natural map gives an isomorphism
$$h^n(\bigvee_{i\in I} X_i) \xrightarrow{\cong} \prod_{i\in I} h^n(X_i).$$
In other words, it takes small coproducts to products.

Remark 4.2. An (un-reduced) generalized cohomology theory is built on good pairs, as opposed to pointed spaces.

There is a standard trick of getting from reduced to un-reduced by adding a disjoint basepoint.

A particular example of generalized cohomology theory is the reduced singular cohomology theory $\tilde{H}^i(-;G)$ for G an abelian group. A nice general construction we can make for any generalized cohomology theory is as follows.

Proposition 4.3 (Puppe Sequence)**.** Suppose we have two cofiber sequence (this is called a **Puppe sequence** sometimes)
$$\underbrace{A \to X \to Q}_{\textbf{cofiber}} \to \underbrace{\Sigma A \to \Sigma X \to \Sigma Q}_{\textbf{cofiber}},$$
then any other consecutive triplets in the sequence is a cofiber sequence.

Proof Sketch. This follows from a problem set we discussed last week. ■

Definition 4.4. Applying $h^n(-)$ now to the Puppe sequence, we get a **long exact sequence** in (reduced) generalized cohomology theories as

$$... \to h^n(Q) \to h^n(X) \to h^n(A) \to h^{n-1}(Q) \to h^{n-1}(X) \to h^{n-1}(A) \to ...$$

4.2 K-theory as an Example

Now we discuss K-theory as an example of generalized cohomology theories.

Definition 4.5. Let X be a **compact Hausdorff** space, we define

$$K^0(X) := \langle \text{isomorphism classes of } \mathbb{C}\text{-vector bundles over } X\rangle / \langle [E] + [E'] = [E \oplus E'] \rangle.$$

In practice, each element in $K^0(X)$ can be non-uniquely written as $[E] - [E']$, for a pair of vector bundles.

Example 4.6. Here are some examples:

1. $K^0(*) \cong \mathbb{Z}$ (classified by the dimension of the vector bundles).
2. $K^0(S^1) \cong \mathbb{Z}$ (which is harder to show).

Theorem 4.7. There is a correspondence between

1. { n-dimensional $\mathbb{C}$-vector bundles over X} mod isomorphisms.
2. Homotopy classes of maps $X \to \mathrm{BU}(n)$.

In particular, if $X \simeq Y$, then $K^0(X) = K^0(Y)$.

This is not yet a (reduced) generalized cohomology theory yet (we only really defined one group).

Example 4.8. K^0 fails the wedge axiom. This is because we can take $X = X \vee *$, and $K^0(X) = K^0(X) \oplus K^0(*) = K^0 \oplus \mathbb{Z}$.

Definition 4.9. We say two vector bundles E, E' over X are stably equivalent if there exists two trivial vector bundles ϵ^i, ϵ^j (where the superscript indicates their dimensions) such that

$$E \oplus \epsilon^i \cong E' \oplus \epsilon^j.$$

From here, we defined the 0-th reduced K-theory as follows.

Definition 4.10. We define $\tilde{K}^0(X)$ as the isomorphism classes of $\mathbb{C}$-vector-bundles over X modulo stable equivalences.

One can take the following fact.

Proposition 4.11. For X a **paracompact space** and a vector bundle $E \to X$, there exists $E' \to X$ such that $E \oplus E'$ is trivial.

It then follows that $\tilde{K}^0$ is an abelian group (with inverses), as follows.

Lemma 4.12. Any element of $K^0(X)$ can be written as a vector bundle minus a trivial bundle.

Proof. If $[E] - [E'] \in K^0(X)$, we find F such that $E' \oplus F = \epsilon^i$. We then have that

$$[E] - [E'] = [E] - [E'] - ([F] - [F]) = [E \oplus F] - [\epsilon^i].$$

■

The previous lemma gives a well-defined surjection.

Definition 4.13. $K^0(X) \to \tilde{K}^0(X)$ given by sending $[E] - [\epsilon^i] \mapsto [E]$. This also shows $\tilde{K}^0(X)$ is abelian.

Note this also gives an exact sequence

$$\mathbb{Z} \to K^0(X) \to \tilde{K}^0(X).$$

Example 4.14. $\tilde{K}^0(*) = 0$.

Finally we define the reduced K-theories as follows.

Definition 4.15. We define $\tilde{K}^i(X) := \tilde{K}^0(\Sigma^i X)$.

One can check the following.

Theorem 4.16. The functors $\tilde{K}^i(-)$ forms a reduced generalized cohomology theory.

4.3 Ω-Spectrum and Infinite Loop Spaces

Definition 4.17. An Ω-spectrum is a sequence $\{X_n, f_n : \Omega X_{n+1} \to X_n\}_{\geq 0}$ where each f_n is an equivalence.

Definition 4.18. An infinite loop space X is a space such that there exists spaces X_i for each $i \geq 0$ such that $X \simeq \Omega^i X_i$.

The definition of an Ω-spectrum implies that every level of an Ω-spectrum is an infinite loop space. A rich source of them comes from the following.

Definition 4.19. Recall for an abelian group G, an **Eilenberg-MacLane space** $K(G,n)$ is a space such that

$$\pi_i(K(G,n)) = \begin{cases} G, i = n \\ 0, i \neq n \end{cases}.$$

This is unique up to weak homotopy equivalence (and are homotopy equivalent if we restrict to CW complexes), and one can show that

$$\Omega K(G, n+1) \simeq K(G, n).$$

We define HG, called the **Eilenberg MacLane spectra** for G as $\{K(G,n)\}_{n\geq 0}$. This is an Ω**-spectrum**, and every $K(G,n)$ is an **infinite loop space**.

We can get generalized cohomology theories out of Ω-spectrum as follows!

Proposition 4.20. Let $X = \{X_n, f_n\}_{n\geq 0}$ be an Ω-spectrum, then the homotopy classes of maps for each n

$$h^n := [-, X_n]$$

is a (reduced) generalized cohomology theory.

Proof. We verify each axioms as follows:

1. For **homotopy invariance**: If $f, f' : Y \to Z$ are homotopic, then for any $g : Z \to X_n$, $g \circ f$ is equivalent to $g \circ f'$. Thus, f and f' induce the same morphism.
2. For **exactness**: Suppose we have a cofibration sequence $A \to Y \to Q$. Then there is a standard fact that the map $f : Y \to X_n$ extends to the cofiber Q, which has an explicit model as $Y \cup CA$, if and only if $f|_A$ is null-homotopic. This makes the sequence
$$[Q, X_n] \to [Y, X_n] \to [A, X_n]$$
exact.
3. For **suspension**: We apply the suspension-loop adjunction and the definition of Ω-spectrum to get:
$$[\Sigma Y, X_{n+1}] = [Y, \Omega X_{n+1}] = [Y, X_n].$$
4. For **wedge axioms**: We want to show that $[\bigvee_{i\in I} Y_i, X_n] \cong \prod_{i\in I}[Y_i, X_n]$. This is true for maps in general, by universal property, because $\bigvee_{i\in I} Y_i \to X_n$ is the same as specifying maps $\{Y_i \to X_i\}_{i\in I}$.

■

4.4 Brown's Representability

In light of Proposition 4.20, one might wonder to what extent the converse is true. The answer is - all of them!

Theorem 4.21 (Brown's Representability Theorem)**.** For any (reduced) generalized cohomology theory $\{h^n\}$, there exists an Ω-spectrum $X = \{X_n\}$ such that

$$h^n = [-, X_n].$$

Example 4.22. The Eilenberg-MacLane spectrum HG is representing $\tilde{H}^*(-, G)$, with $\tilde{H}^n(X; G) = [X, K(G,n)]$.

We now pay our attention to reduced (complex) K-theory and ask what spectrum represents them.

Definition 4.23. Let U be the colimit of $U(n)$, the unitary matrices, over n.

Each map $U_n \to U_{n+1}$ is given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.

We define BU as the classifying space of U, with the Euclidean topology imposed on U. Alternatively, BU is the homotopy colimit of the $BU(n)$'s.

To construct the spectrum, we state the following theorem without proof.

Theorem 4.24 (Bott Periodicity over $\mathbb{C}$)**.** There is an equivalence $U \simeq \Omega^2 U$. Furthermore, $\Omega U \simeq \mathbb{Z} \times \mathrm{BU}$.

Theorem 4.25. The sequence

$$\mathbb{Z} \times \mathrm{BU}, U, \mathbb{Z} \times \mathrm{BU}, \ldots$$

with natural equivalences outlined above is an Ω-spectrum, called KU. In fact, this represents (reduced) complex K-theory.

Example 4.26. In light of the discussion for reduced vs unreduced, we have that

$$K(X) = [X \sqcup \{*\}, \mathrm{BU} \times \mathbb{Z}] \text{ and } \tilde{K}(X) = [X, \mathrm{BU} \times \mathbb{Z}].$$

4.5 Sequential Spectrum

There is also a definition of spectrum using suspensions.

Definition 4.27. A sequential spectrum is a sequence $X = \{X_n, f_n : \Sigma X_n \to X_{n+1}\}$, with no additional assumption on f_n.

Example 4.28. The sphere spectrum is $\mathbb{S} = \{S^i\}$ for $i \geq 0$. Every Ω-spectrum is a suspension spectrum.

Definition 4.29. Given a suspension spectrum $X = \{X_n\}_{n \geq 0}$, the stable homotopy groups of X are

$$\pi_n(X) := \mathrm{colim}_{i \to \infty} \pi_{i+n}(X_i).$$

Importantly, here n can be negative.

Example 4.30. We have that $\pi_0(HG) = G$ and $\pi_n(HG) = 0$ if $n \neq 0$.

Just like reduced generalized cohomology theory, we also have an analog of "reduced generalized homology theory" with $h_n(-)$. There is a dual of Brown's representability as follows.

Theorem 4.31. For every h_n, there exists a sequential spectrum X such that:

1. $h_n(*) \cong \pi_n(X)$.

2. $h_n(Y) \cong \pi_n(X \wedge Y)$.

5 Week 5: Sequential Spectra by Sam Richter

The outline of this talk is as follows:

1. We will first give some motivation on why we are discussing sequential spectra.
2. Then we will define sequential spectra and give examples.
3. We will then revisit Ω-spectra.
4. Finally, we will discuss some operations on spectra.

5.1 Motivation for Sequential Spectra

Two of the big theorems we have seen so far are:

1. The Freudenthal Suspension Theorem.
2. Brown Representability Theorem.

Recall a particular corollary of the Freudenthal Suspension Theorem is given by:

Corollary 5.1. The sequence

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \ldots$$

is eventually constant.

The upshot of this corollary is - if we only care about stability, we should care about sequences of spaces

$$X, \Sigma X, \Sigma^2 X, \ldots$$

On the other hand, Brown Representability Theorem tells us we should care about the sequence of spaces $\{X_n\}_{n\geq 0}$ with $X_n \xrightarrow{\sim} \Omega X_{n+1}$.

Finally, we know there is an adjunction between Σ and Ω, so we might expect the two to be related.

5.2 Spectra

Definition 5.2. A (sequential) spectrum X is a sequence of based spaces $\{X_n\}_{n\geq 0}$ and bonding maps $\xi_n : \Sigma X_n \to X_{n+1}$. A **map of spectra** $f : X \to Y$ is a sequence of continuous based maps $f_n : X_n \to Y_n$ such that the following diagram commutes:

$$\begin{tikzcd}
\Sigma X_n \arrow[r, "\xi_n"] \arrow[d, "f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\
\Sigma Y_n \arrow[r, "\xi'_n"] & Y_{n+1}
\end{tikzcd}$$

This forms a category of spectra which we denote as Sp.

Remark 5.3. This is sometimes also called a pre-spectra, to distinguish from other kinds of spectra. For this talk we will just stick with spectra.

Furthermore, due to the adjunction, we could equivalently defined the bonding maps as $\xi_n : X_n \to \Omega X_{n+1}$.

Definition 5.4. If X is a spectrum, for $k \in \mathbb{Z}$. The k-th homotopy group $\pi_k(X)$ is the colimit of the following sequence

$$\ldots \to \pi_{k+n}(X_n) \xrightarrow{\phi} \pi_{k+1+n}(X_{n+1}) \to \ldots$$

Here ϕ is given by applying the suspension functor,

$$[S^{k+n}, X_n] \xrightarrow{\sigma} [\Sigma S^{k+n} = S^{k+n+1}, \Sigma X_n],$$

and then composing with ξ_n:

$$[S^{k+n+1}, \Sigma X_n] \to [S^{k+n+1}, X_{n+1}]$$

When k is negative, we just say that the system is defined when $n \geq |k|$. This gives us a way to talk about negative homotopy groups!

Definition 5.5. A **stable equivalence** (or a π_*-isomorphism) is a map of spectra $f : X \to Y$ inducing an isomorphism $\pi_k(X) \xrightarrow{\cong} \pi_k(Y)$ for all $k \in \mathbb{Z}$.

Remark 5.6. Stable equivalences in Sp satisfy the 2-out-of-6 property. Thus, we can define an associated homotopy category $\mathrm{Ho}(\mathrm{Sp})$. This is called the **stable homotopy category**.

Here are some examples of sequential spectra.

Example 5.7. Let X be a based space, we can form a **suspension spectrum** $\Sigma^\infty X$ which at level n is $\Sigma^n X$. The bonding map is given by

$$X \wedge S^n \wedge S^1 \xrightarrow{\mathrm{id} \wedge h} X \wedge S^{n+1},$$

where $h : S^n \wedge S^1 \to S^{n+1}$ is the canonical identification (see Definition 2.1.5. of [Mal26]).

Let B be an un-based space, we can form a **suspension spectrum** $\Sigma^\infty_+ B$ defined as

$$\Sigma^\infty_+ B := \Sigma^\infty(B_+), B_+ = B \sqcup \{*\}.$$

B_+ is the disjoint union of B with a point, thought of as a formal basepoint.

Remark 5.8. The homotopy groups of $\Sigma^\infty X$ are exactly the stable homotopy groups of X.

Example 5.9. The suspension spectrum of S^0 (two points) is called the **sphere spectrum**, denoted $\mathbb{S}$.

Example 5.10. The zero spectrum $*$ is the suspension spectrum of a point. Note that this is the zero object in X! Every spectrum X has unique maps $* \to X$ and $X \to *$.

The existence of this zero object allows us to define.

Definition 5.11. Given two spectra X, Y, the zero map $X \to Y$ is the unique map of spectra that factors through the zero spectrum, ie. it is the composition $X \to * \to Y$.

Here is another example, which we saw as an Ω-spectrum last time.

Example 5.12. Let A be an abelian group. We define the **Eilenberg-MacLane spectrum** HA where $(HA)_n := K(A, n)$. We define the spectrum using the adjoint bonding maps, which are the weak homotopy equivalences

$$K(A, n) \xrightarrow{\sim} \Omega K(A, n+1).$$

In particular, we have that

$$\pi_i(HA) = \begin{cases} A, i = 0 \\ 0, i \neq 0 \end{cases}.$$

The last example, which we also saw last time, is the complex K-theory spectrum KU.

Example 5.13. KU is the sequence

$$\mathbb{Z} \times \mathrm{BU}, \mathrm{U}, \mathbb{Z} \times \mathrm{BU}, \mathrm{U}, \ldots.$$

where the equivalences are given by the adjoint bonding maps described as in the last lecture. In this case, we have that

$$\pi_i(\mathrm{KU}) = \begin{cases} \mathbb{Z}, & \text{if i is even} \\ 0, & \text{if i is odd} \end{cases}.$$

Warning: The bonding maps matter a lot! Let's suppose we construct a spectrum X where $X_n = S^n$, but the bonding maps

$$\Sigma S^n \to S^{n+1} \text{ is the constant map to the basepoint.}$$

It turns out in this case, X is stably equivalent to $*$, the zero spectrum!

5.3 Revisiting Ω-Spectra

Let us recall the definition of an Ω-spectrum from last time.

Definition 5.14. An Ω-spectrum is a spectrum X in which all of the adjoint bonding maps $X_n \to \Omega X_{n+1}$ are weak homotopy equivalences.

Non-Example: The sphere spectrum, in particular, is not an Ω-spectrum.

Definition 5.15. A map $f : X \to Y$ of spectra is a level equivalence if $f_n : X_n \to Y_n$ is a weak homotopy equivalence for all n.

Remark 5.16. Every stable equivalence between Ω-spectra is actually a **level equivalence** (which means weak homotopy equivalences on each level).

Proposition 5.17. To each spectrum X_i, we may assign an Ω-spectrum RX and a stable equivalence $X \to RX$, in a natural way. This RX is known as the **fibrant replacement** of X. This is natural in the sense that there is a natural transformation $\mathrm{Id}_{\mathrm{Sp}} \to R$.

Proof Sketch. Here we only construct the spectrum RX. We define

$$(RX)_n := \operatorname{hocolim}(X_n \xrightarrow{\tilde{\xi}_n} \Omega X_{n+1} \xrightarrow{\Omega\tilde{\xi}_{n+1}} \Omega^2 X_{n+2} \to \ldots)$$

where the $\tilde{\xi}_n$ is the adjoint bonding map.

Question 5.18. Why is this an Ω-spectrum?

Indeed, we see that

$$(RX)_n = \operatorname{hocolim}(X_n \xrightarrow{\tilde{\xi}_n} \Omega X_{n+1} \xrightarrow{\Omega\tilde{\xi}_{n+1}} \Omega^2 X_{n+2} \to \ldots)$$

is equivalent, by cofinality, to

$$\operatorname{hocolim}_{m\to\infty}(\Omega^{1+m} X_{n+1+m}).$$

This cofinality condition holds because we recall $\operatorname{hocolim} = \operatorname{colim} \circ Q$, where Q is an appropriate left deformation.

Now there is a natural identification (which is Exercise 1.7.32 of [Mal26])

$$\operatorname{hocolim}_{m\to\infty}(\Omega^{1+m} X_{n+1+m}) \simeq \Omega(\operatorname{hocolim}_{m\to\infty} \Omega^m X_{n+1+m}) = \Omega(RX)_{n+1}.$$

Thus, this in fact is an Ω-spectrum. ∎

Definition 5.19. The infinite loop space $\Omega^\infty X$ is the 0-th space of any Ω-Spectrum RX receiving a stable equivalence from X. This in particular gives a functor

$$\Omega^\infty : \mathrm{Sp} \to \mathrm{Top}_* .$$

The following is an exercise that will appear in this week's problem sessions.

Proposition 5.20. Ω^∞ is right adjoint to Σ^∞ in $\mathrm{Ho}(\mathrm{Sp})$.

Definition 5.21. We write $QX := \operatorname{colim}(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \ldots)$.

Proposition 5.22. $QX = \Omega^\infty\Sigma^\infty X$.

Proof. Given a based space X, we note $R\Sigma^\infty X$ have levels given by:

$$(R\Sigma^\infty X)_n = \operatorname{colim}(\Sigma^n X \to \Omega\Sigma^{n+1} X \to \Omega^2\Sigma^{n+2} \ldots).$$

In this case, we have

$$\operatorname{colim}(\Omega^m\Sigma^{n+m} X) \xrightarrow{\cong} \operatorname{colim}(\Omega^{1+m}\Sigma^{n+1+m} X) \xrightarrow{\cong} \Omega \operatorname{colim}(\Omega^m\Sigma^{n+1+m} X),$$

where the first isomorphism is given by cofinality and the second is given by Exercise 1.7.31 of [Mal26].

In particular, we see that

$$\Omega^\infty\Sigma^\infty X = (R\Sigma^\infty X)_0 = \operatorname{colim}(X \to \Omega\Sigma X \to \Omega^2\Sigma^2 X \to \ldots) = QX.$$

∎

5.4 Operations on Spectra

In this part, we fix X and Y to be spectra.

Definition 5.23. The **wedge sum** $X \vee Y$ is the spectrum whose n-th level is $(X \vee Y)_n := X_n \vee Y_n$. We define the bonding maps to be

$$\Sigma(X_n \vee Y_n) \xrightarrow{\cong} (\Sigma X_n) \vee (\Sigma Y_n) \to X_{n+1} \vee Y_{n+1}.$$

The first isomorphism is given because suspension is left adjoint and hence commutes with coproducts.

Definition 5.24. The **product** $X \times Y$ is the spectrum whose n-th level $(X \times Y)_n := X_n \times Y_n$. The bonding maps are given by

$$\Sigma(X_n \times Y_n) \to (\Sigma X_n) \times (\Sigma Y_n) \to X_n \times Y_n,$$

where the first map is given by applying the functor $\Sigma(\bullet)$ to projection onto each component respectively.

Example 5.25. Let A, B be based spaces, then:

$$\Sigma^\infty A \vee \Sigma^\infty B \simeq \Sigma^\infty (A \vee B).$$

Let G_1, G_2 be abelian groups, then

$$HG_1 \times HG_2 \cong H(G_1 \times G_2).$$

Definition 5.26. For an $d \in \mathbb{Z}$, we can define the d-fold shift operator which sends X to $\operatorname{sh}^d(X)$ by

$$(\operatorname{sh}^d X)_n = X_{d+n},$$

with the same bonding maps. If $n + d < 0$, we just declare that $(\operatorname{sh}^d X)_n = *$.

Remark 5.27. Observe that

$$\pi_k(\operatorname{sh}^d X) = \pi_{k-d}(X).$$

Now we discuss a notion of tensoring a spectrum with a space.

Definition 5.28. Let K be a based space, the **tensor** (or **smash**) product $K \wedge X$ is given by smasking K with every level of X. In other words,

$$(K \wedge X)_n = K \wedge X_n.$$

The bonding map is given by

$$K \wedge X_n \wedge S^1 \xrightarrow{\operatorname{id} \wedge \xi_n} K \wedge X_{n+1}.$$

Example 5.29. For a spectrum X, the **reduced suspension** is $\Sigma X := S^1 \wedge X$.

Example 5.30. For a based space K, we have that $\Sigma^\infty K \simeq K \wedge \mathbb{S}$.

Opposite to the notion of tensors, we also discuss a notion of cotensor operation.

Definition 5.31. For a based space K, the **cotensor** or **function spectrum** $F(K, X)$ is the spectrum whose n-th level is given by

$$(F(K, X))_n = \operatorname{Map}_*(K, X_n).$$

The adjoint bonding maps in this case is given by

$$\operatorname{Map}_*(K, X_n) \xrightarrow{\operatorname{Map}_*(K, \tilde{\xi_n})} \operatorname{Map}_*(K, \Omega X_{n+1}) \xrightarrow{\cong} \Omega \operatorname{Map}_*(K, X_{n+1}).$$

To explain the second isomorphism, we recall that

$$\operatorname{Map}_*(K, \Omega X_{n+1}) \coloneqq \operatorname{Map}_*(K, \operatorname{Map}_*(S^1, X_{n+1})) \cong_{\text{tensor-hom adjunction}} \operatorname{Map}_*(K \wedge S^1, X_{n+1})$$

$$\cong \operatorname{Map}_*(S^1 \wedge K, X_{n+1}) \cong_{\text{tensor-hom adjunction}} \operatorname{Map}_*(S^1, \operatorname{Map}_*(K, X_{n+1})) = \Omega \operatorname{Map}_*(K, X_{n+1}).$$

Example 5.32. For a spectrum X, we define $\Omega X \coloneqq F(S^1, X)$.

Remark 5.33. There is an adjunction $\Sigma \dashv \Omega$ on Sp. In general, $K \wedge -$ and $F(K, -)$ are adjoiint functors on Sp.

Finally, we discuss two operations that will lead to next week's talk.

Definition 5.34. The **homotopy pullback** $X \times_B^h Y$ of a diagram of spectra

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & B \end{tikzcd}$$

is at eahc level the homotopy fiber product

$$(X \times_B^h Y)_n = X_n \times_{B_n}^h Y_n,$$

the homotopy pullback on the level of spaces. Just to recall, this is exactly modeled as

$$X_n \times_{B_n}^h Y_n = X_n \times_{B_n} B_n^I \times_{B_n} Y_n.$$

Example 5.35. Let $f : X \to Y$ be a map of spectra, the homotopy pullback $* \times_Y hX$ is called the **homotopy fiber** Ff. Explicitly

$$(Ff)_n = F(f_n) = X_n \times_{Y_n} \operatorname{Map}_*(I, Y_n).$$

Here $F(f_n)$ denotes the homotopy fiber on spaces.

Definition 5.36. The **homotopy pushout** $X \cup_A^h Y$ of a diagram of spectra

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \\ Y & \end{tikzcd}$$

is at each level the based mapping cylinder

$$(X \cup_A^h Y)_n = X_n \cup_A^n Y_n$$

with level wise homotopy pushout.

Example 5.37. Let $f : X \to Y$ be a map of spectra, the homotopy pushout $* \cup_X^h Y$ is called the homotopy cofiber Cf. Here $(Cf)_n = C(f_n)$ is the homotopy cofiber on spaces.

Remark 5.38. More generally, we may define a homotopy colimit of a diagram of spectra by taking the homotopy colimit level wise on each space. This is Definition 2.3.17 of [Mal26].

5.5 Problem Session Addendum and Remarks

We record the following exercise from the problem sessions which will be useful later.

Lemma 5.39. Consider a sequence of spectra with maps

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to \ldots$$

then for any $k \in \mathbb{Z}$,

$$\pi_k(\mathrm{hocolim}_i\, X_i) \cong \mathrm{colim}_i\, \pi_k(X_i).$$

The following fact is also stated in the exercises that will be very helpful.

Proposition 5.40. Let X be a sequential spectrum such that $\pi_i(X) = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

Corollary 5.41. Define the rationalization of the sphere spectrum $\mathbb{S}$ as

$$\mathbb{S}_{\mathbb{Q}} := \mathrm{hocolim}(\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \xrightarrow{\cdot 3} \mathbb{S} \xrightarrow{\cdot 4} \ldots)$$

Then it follows that $\mathbb{S}_{\mathbb{Q}}$ is equivalent to $H\mathbb{Q}$.

Proof. Serre's finiteness theorem tells us that the higher homotopy groups of $\mathbb{S}$ above 0 are all finite and hence rationalize to 0. The result then follows as $\pi_0(\mathbb{S}) = \mathbb{Z}$. ■

Warning! We end the discussion here with a cautionary example.

Example 5.42. It may be tempting to think that for based spaces X and Y that are not homotopy equivalent, we also have $\Sigma^\infty X \not\simeq \Sigma^\infty Y$. This is however not true. For example, we know that $\pi_3(S^2) = \mathbb{Z}$, consider $X = S^2 \cup_0 D^4$ being given by the degree 0 attachement map and $Y = S^2 \cup_{12} D^4$ being given by the degree 12 attachment map. Then unstably X and Y are not homotopy equivalent, but stably since $\pi_1(\mathbb{S}) = \mathbb{Z}/2$, the two spaces would give equivalent spectra.

6 Week 6: Stability Theorems by Sergio Lenis, Expanded by Mattie Ji

Slogan: Spectra is a better category than spaces for stable phenonmena.

Recall the category of spectra Sp are given by objects being a collection $\{X_n\}_{n\geq 0}$ along with bonding maps $\xi_n : \Sigma X_n \to X_{n+1}$. A morphism of spectra $f : X \to Y$ is a collection of maps $f_n : X_n \to Y_n$ that commutes with the bonding maps.

Definition 6.1. A map $f : X \to Y$ is a **stable equivalence** if for all $k \in \mathbb{Z}$, the induced map $f_* : \pi_k(X) \to \pi_k(Y)$.

Remark 6.2. As noted from last time, stable equivalences satisfy the 2 out of 6 properties, so we can form a homotopy category.

The outline of today's talk is as follows:

1. Σ and Ω are inverses of each other.
2. Cofiber and Fiber sequences are the same.
3. Homotopy pullbacks and Homotopy pushouts are the same.

... in the homotopy category $\mathrm{Ho}(\mathrm{Sp})$ (i.e., up to stable equivalence).

6.1 Stability as an Equivalence $\Sigma^{-1} \simeq \Omega$

Recall that for a spectra X with bonding maps $\xi_n : \Sigma X_n \to X_{n+1}$ and adjoint bonding maps $\tilde{\xi_n}$, we defined:

1. $\Sigma X = \{S^1 \wedge X_n\}_{n\geq 0}$ with bonding maps
$$\Sigma(\Sigma X)_n = S^1 \wedge X_n \wedge S^1 \xrightarrow{id\wedge\xi_n} S^1 \wedge X_{n+1} = (\Sigma X)_{n+1}.$$
Note here Σ on a space **smashes with S^1 on the right**.
2. $\Omega X = \{\mathrm{Map}_*(S^1, X_n)\}_{n\geq 0}$ with adjoint bonding maps
$$(\Omega X)_n = \mathrm{Map}_*(S^1, X_n) \xrightarrow{\tilde{\xi_n}\circ -} \mathrm{Map}_*(S^1, \Omega X_{n+1}) \cong \Omega\,\mathrm{Map}_*(S^1, X_{n+1}) = \Omega(\Omega X)_{n+1}.$$

Recall from last time, we have:

Theorem 6.3. There is an adjunction
$$\Sigma : \mathrm{Sp} \rightleftarrows \mathrm{Sp} : \Omega.$$

Let $X \to \Omega\Sigma X$ be the unit and the counit $\Sigma\Omega X \to X$. We first prove the following result.

Proposition 6.4 (Proposition 2.4.1 of [Mal26]). For any $X \in \mathrm{Sp}$, we have natural isomorphisms
$$\pi_{k+1}(\Sigma X) \cong \pi_k(X) \cong \pi_{k-1}(\Omega X).$$

Proof. We have the (left side) the colimit diagram for $\pi_k(X)$ and the (right side) the colimit diagram for $\pi_{k+1}(\Sigma X)$ given by

$$\begin{tikzcd} \vdots \arrow[d] \\ \pi_{k+2n}(X_{2n}) \arrow[d] \\ \pi_{k+2n+1}(X_{2n} \wedge S^1) \arrow[d] \\ \pi_{k+2n+1}(X_{2n+1}) \arrow[d] \\ \pi_{k+2n+2}(X_{2n+1} \wedge S^1) \arrow[d] \\ \pi_{k+2n+2}(X_{2n+2}) \arrow[d] \\ \pi_{k+2n+3}(X_{2n+2} \wedge S^1) \arrow[d] \\ \vdots \end{tikzcd} \qquad \begin{tikzcd} \vdots \arrow[d] \\ \pi_{(k+1)+2n}(S^1 \wedge X_{2n} = (\Sigma X)_{2n}) \arrow[d] \\ \pi_{(k+1)+2n+1}(S^1 \wedge X_{2n} \wedge S^1) \arrow[d] \\ \pi_{(k+1)+2n+1}(S^1 \wedge X_{2n+1} = (\Sigma X)_{2n+1}) \arrow[d] \\ \pi_{(k+1)+2n+2}(S^1 \wedge X_{2n+1} \wedge S^1) \arrow[d] \\ \pi_{(k+1)+2n+2}(S^1 \wedge X_{2n+2} = (\Sigma X)_{2n+2}) \arrow[d] \\ \vdots \end{tikzcd}$$

By cofinality, we can restrict to terms of the form $\pi_{k+2n+1}(\Sigma(X_{2n})) = \pi_{k+2n+1}(X_{2n} \wedge S^1)$ on the left side (recall $\Sigma(X_{2n})$ are $X_{2n} \wedge S^1$, and $(\Sigma X)_{2n} = S^1 \wedge X_{2n}$). We can also restrict the terms on the right side to those of the form $\pi_{(k+1)+2n}(S^1 \wedge X_{2n}) = \pi_{k+1+(2n)}((\Sigma X)_{2n})$. The two sides now takes the form of:

$$\begin{tikzcd} \vdots \arrow[d] \\ \pi_{k+2n+1}(X_{2n} \wedge S^1) \arrow[d] \\ \pi_{k+2n+2}(X_{2n+1} \wedge S^1) \arrow[d] \\ \pi_{k+2n+3}(X_{2n+2} \wedge S^1) \arrow[d] \\ \vdots \end{tikzcd} \qquad \begin{tikzcd} \vdots \arrow[d] \\ \pi_{(k+1)+2n}(S^1 \wedge X_{2n} = (\Sigma X)_{2n}) \arrow[d] \\ \pi_{(k+1)+2n+1}(S^1 \wedge X_{2n+1} = (\Sigma X)_{2n+1}) \arrow[d] \\ \pi_{(k+1)+2n+2}(S^1 \wedge X_{2n+2} = (\Sigma X)_{2n+2}) \arrow[d] \\ \vdots \end{tikzcd}$$

where the arrows are implicitly compositions of maps from the previous colimit.

We claim now that their colimits are isomorphic. There are some details to be checked here, the swap homeomorphism $X_{2n} \wedge S^1 \to S^1 \wedge X_{2n}$ induces an isomorphism $\pi_{k+2n+1}(X_{2n} \wedge S^1) \to \pi_{(k+1)+2n}(S^1 \wedge X_{2n} = (\Sigma X)_{2n})$, but we need to check this isomorphism is compatible with the colimit maps on both sides. Actually, we will check that the following diagram commutes instead, which will show the isomorphism due to cofinality:

$$\begin{tikzcd}
\pi_{k+2n+1}(X_{2n} \wedge S^1) \arrow[r, "\text{swap}"] \arrow[d] & \pi_{(k+1)+2n}(S^1 \wedge X_{2n} = (\Sigma X)_{2n}) \arrow[d] \\
\pi_{k+2n+2}(X_{2n+1} \wedge S^1) \arrow[d] & \pi_{(k+1)+2n+1}(S^1 \wedge X_{2n+1} = (\Sigma X)_{2n+1}) \arrow[d] \\
\pi_{k+2n+3}(X_{2n+2} \wedge S^1) \arrow[r, "\text{swap}"] & \pi_{(k+1)+2n+2}(S^1 \wedge X_{2n+2} = (\Sigma X)_{2n+2})
\end{tikzcd}.$$

Suppose $\alpha \in \pi_{k+2n+1}(X_{2n} \wedge S^1)$, which is given by $\alpha : S^{k+2n+1} \to X_{2n} \wedge S^1$. Going down first and left, the composition follows by:

$$\underbrace{\alpha \mapsto \xi_{2n} \circ \alpha \mapsto (\xi_{2n} \circ \alpha \wedge \mathrm{id}_{S^1})}_{\text{first map}} \mapsto \underbrace{\xi_{2n+1} \circ (\xi_{2n} \circ \alpha \wedge \mathrm{id}_{S^1}) \mapsto (\xi_{2n+1} \circ (\xi_{2n} \circ \alpha \wedge \mathrm{id}_{S^1})) \wedge \mathrm{id}_{S^1}}_{\text{second map}} \mapsto \text{ add swap.}$$

Going right first and then down, the compoisiton follows by

$$\alpha \mapsto \text{swap} \circ \alpha \mapsto \underbrace{(\text{swap} \circ \alpha) \wedge \mathrm{id}_{S^1} \mapsto (\mathrm{id}_{S^1} \wedge \xi_{2n}) \circ ((\text{swap} \circ \alpha) \wedge \mathrm{id}_{S^1})}_{\text{first map}} \mapsto$$

$$\underbrace{(\mathrm{id}_{S^1} \wedge \xi_{2n}) \circ ((\text{swap} \circ \alpha) \wedge \mathrm{id}_{S^1}) \wedge \mathrm{id}_{S^1} \mapsto (\mathrm{id}_{S^1} \wedge \xi_{2n+1}) \circ (\mathrm{id}_{S^1} \wedge \xi_{2n}) \circ ((\text{swap} \circ \alpha) \wedge \mathrm{id}_{S^1}) \wedge \mathrm{id}_{S^1}}_{\text{second map}}.$$

Thus, we see that commutativity is equivalent to checking that the following diagram is **homotopy commutative**:

$$\begin{tikzcd}
X_{2n} \wedge S^1 \wedge S^1 \wedge S^1 \arrow[r, "\xi_{2n} \wedge \mathrm{id} \wedge \mathrm{id}"'] \arrow[d, "\cong"'] & X_{2n+1} \wedge S^1 \wedge S^1 \arrow[r, "\xi_{2n+1} \wedge \mathrm{id}"] & X_{2n+2} \wedge S^1 \arrow[d, "\cong"] \\
S^1 \wedge X_{2n} \wedge S^1 \wedge S^1 \arrow[r, "\mathrm{id} \wedge \xi_{2n} \wedge \mathrm{id}"'] & S^1 \wedge X_{2n+1} \wedge S^1 \arrow[r, "\mathrm{id} \wedge \xi_{2n+1}"'] & S^1 \wedge X_{2n+2}
\end{tikzcd}$$

Observe first that this diagram is not strictly commutative, because going right and down preserves the last copy of S^1 and going down and left preserves the first copy of S^1 (counting from left to right). The other S^1-factors had an effect too, we just pointed out one reason why it is not strictly commutative.

However, the two maps differs from a self-map of $S^1 \wedge S^1 \wedge S^1 \simeq S^3$ given by permuting the S^1's (call them a, b, c) with

$$a \mapsto b, b \mapsto c, c \mapsto a.$$

We claim this map is homotopic to the identity on S^3, which will show the diagram is homotopy commutative and show that $\pi_{k+1}(\Sigma X) \cong \pi_k(X)$. This seems intuitive, but it also follows from a general lemma (see Lemma A.2 in the appendix).

For $\pi_k(X) \cong \pi_{k-1}(\Omega X)$, note the stable homotopy group can be alternatively given by

$$\pi_k(X) = \operatorname{colim}(\dots \to \pi_{k+n}(X_n) \xrightarrow{\tilde{\xi}_n} \pi_{k+n}(\Omega X_{n+1}) \xrightarrow{\cong} \pi_{k+n+1}(X_{n+1}) \to \dots).$$

The proof then follows from a similar comparison of colimits and a similar square. ■

Remark 6.5. A similar proof also shows that $\Sigma X \simeq \operatorname{sh} X$.

Corollary 6.6. A map $f : X \to Y$ is a stable equivalence if and only if $\Sigma f : \Sigma X \to \Sigma Y$ is a stable equivalence, if and only if $\Omega f : \Omega X \to \Omega Y$ is a stable equivalence.

Proof. Clearly the isomorphisms constructed in the proofs above are natural. In other words, the map $f : X \to Y$ induces a commutative diagram:

$$\begin{tikzcd} \pi_{k+1}(\Sigma X) \arrow[r, "\cong", dash] \arrow[d, "\Sigma f_*"'] & \pi_k(X) \arrow[r, "\cong", dash] \arrow[d, "f_*"] & \pi_{k-1}(\Omega X) \arrow[d, "\Omega f_*"] \\ \pi_{k+1}(\Sigma Y) \arrow[r, "\cong"', dash] & \pi_k(Y) \arrow[r, "\cong"', dash] & \pi_{k-1}(\Omega Y) \end{tikzcd} .$$

Thus we see one of the vertical arrows is an isomorphism if and only the other two are. ■

Corollary 6.7. The unit and counit maps

$$X \to \Omega\Sigma X, \Sigma\Omega X \to X$$

are stable equivalences.

Proof. Note for all k, there is a commutative square

$$\begin{tikzcd} \pi_k(X) \arrow[r] \arrow[d, "\cong"'] & \pi_k(\Omega\Sigma X) \\ \pi_{k+1}(\Sigma X) \arrow[r, "\cong"'] & \pi_{(k+1)-1}(\Omega\Sigma X) \arrow[u, "="'] \end{tikzcd}$$

and the top vertical arrow can be verified to be the one induced by the unit map. Hence, the unit map is a stable equivalence. The proof for the counit map follows similarly. ■

The **upshot** is that Σ, Ω are inverses up to **stable equivalences**.

6.2 Stability in Terms of (Co)fiber Sequences

Recall for $f : X \to Y$, a cofiber sequence is

$$X \xrightarrow{f} Y \to Cf, \quad (Cf)_n = CX_n \cup_{X_n} Y_n.$$

Similarly, for $g : Y \to Z$, a fiber sequence is

$$Fg \to Y \xrightarrow{g} Z,$$

where $(Fg)_n = Fg_n = Y_n \times_{Z_n} PZ_n$ (PZ_n is the path space).

We may wish to describe both sequences more ambiently, as follows:

Definition 6.8. We define the tuple $(X, Y, Z, f : X \to Y, g : Y \to Z, h : I_+ \wedge X \to Z)$ where h is a homotopy from the zero map to $g \circ f$ (Definition 2.3.11 of [Mal26]), this is just saying there is a collection of based homotopies $h_n : I_+ \wedge X_n \to Z_n$ from constant to $g_n \circ f_n$ that are compatible with bonding maps.

Since h_n is a null-homotopy, it extends to a map $CX_n \to Z_n$. The cofiber Cf_n can be described as the pushout, and there is a natural map:

$$\begin{tikzcd}
X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d] \arrow[ddr, "g_n"] & \\
CX_n \arrow[r] \arrow[drr, "h_n"'] & Cf_n \arrow[dr, dashed, "h_n \cup g_n"] & \\
& & Z_n
\end{tikzcd}$$

We say the data (X, Y, Z, f, g, h) describes a cofiber sequence if the induced map $Cf \to Z$ is a stable equivalence.

The fiber Fg_n can be described as the pullback, and there is natural map:

$$\begin{tikzcd}
X_n \arrow[drr, bend left, "h_n^*"] \arrow[dr, dashed, "f_n \times h_n"] \arrow[ddr, bend right, "f_n"'] & & \\
& Fg_n \arrow[r] \arrow[d] & PZ_n \arrow[d, "\mathrm{ev}_1"] \\
& Y_n \arrow[r, "g_n"'] & Z_n
\end{tikzcd}$$

We say the data (X, Y, Z, f, g, h) describes a fiber sequence if the induced map $X \to Fg$ is a stable equivalence. Note here h_n induces a map $X_n \to PZ_n$ because $g_n \circ f_n$ is null-homotopic (so at $t = 0$ it is constant, which agrees with the definition of PZ_n).

Lemma 6.9 (Lemma 2.4.10-11 of [Mal26])**.** For each fiber or cofiber sequence of spectra $X \to Y \to Z$, we have a LES

$$\ldots \to \pi_k(X) \to \pi_k(Y) \to \pi_k(Z) \to \pi_{k-1}(X) \to \ldots$$

Proof. The proof for **fiber sequence** follows from (a) the same result for spaces and (b) sequential colimits preserves exactness. To be more specific, we know that for a fiber sequence of spaces $X_n \to Y_n \to Z_n$, the following is exact in the middle

$$\pi_{k+n}(X_n) \to \pi_{k+n}(Y_n) \to \pi_{k+n}(Z_n).$$

Taking a sequential colimit in n shows that

$$\pi_k(X) \to \pi_k(Y) \to \pi_k(Z)$$

is **exact in the middle**. To show exactness at $\pi_k(X)$, we consider the following fiber sequence

$$\Omega Z \to X \xrightarrow{f} Y$$

which we know is exact in the middle by the previous deduction and $\pi_k(\Omega Z) \cong \pi_{k+1}(Z)$ by Proposition 6.4. (To be clear, here we take the homotopy fiber of $f : X \to Y$, which can be identified as ΩZ, similar to the statement about spaces). To show exactness at Z, we use π_{k-1} at the fiber sequence

$$\Omega Y \to \Omega Z \to X$$

and Proposition 6.4 again. This verifies the case for fiber sequences.

The statement for **cofiber sequences** is just not true on space, which we will not try to verify.

Indeed, in this case we have that the natural map $Cf \to Z$ is an isomorphism. We wish to show that for all $k \in \mathbb{Z}$, the natural map

$$\pi_k(X) \xrightarrow{f_*} \pi_k(Y) \xrightarrow{g_*} \pi_k(Cf)$$

is exact at $\pi_k(Y)$. It also suffices to verify this at the middle, since we can do a similar trick of extending the cofiber sequence as well.

To show $\mathrm{im}(f_*) \subseteq \ker(g_*)$, we note an element in $\pi_k(X)$ is a map $\alpha : S^{k+n} \to X_n$, and consider the composition

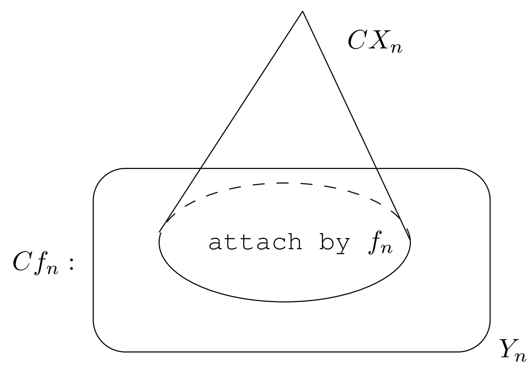
$$S^{k+n} \xrightarrow{\alpha} X_n \xrightarrow{f_n} Y_n \xhookrightarrow{g_n} Cf_n.$$

$$t \mapsto \alpha(t) \mapsto f_n(\alpha(t)) \in Cf_n.$$

To explain the last step, Cf_n can be explicitly built as attaching the cone of X_n to Y_n via the map f_n, so in particular Y_n embeds into Cf_n. g_n can be identified with this map due to our conditions earlier.

But also, $f_n \circ \alpha : S^{k+n}$ lands in the base of the cone CX_n, which is attached to Y_n in Cf_n. This particular we can shift the $f_n \circ \alpha$ up along the cone to the cone point as a null-homotopy. This implies that $g_n \circ f_n \circ \alpha$ is null-homotopic, which verifies the proof.

Here is a conceptual picture that may help:



To show that $\ker(g_*) \subseteq \mathrm{im}(f_*)$, we suppose $\alpha \in \pi_k(Y)$ such that $g_*(\alpha) = 0 \in \pi_k(Z)$. This means that for some n sufficiently large, there is $\alpha_n \in \pi_{k+n}(Y_n)$ such that $(g_n)_*(\alpha_n) = 0 \in \pi_{k+n}(Cf_n)$. In other words, the map

$$S^{k+n} \xrightarrow{\alpha_n} Y_n \xrightarrow{g_n} Cf_n \text{ is null-homotopic.}$$

Since the map $g_n \circ \alpha_n$ is null-homotopic, it lifts to a map through the cone

$$CS^{k+n} \xrightarrow{\bar{\alpha_n}} Cf_n.$$

Now recall a sphere S^{N+1} is two pieces of CS^N glued along their common base S^N. We can use this to define a map

$$\beta'_{n+1} : S^{k+n+1} \xrightarrow{\cong} CS^{k+n} \cup_{S^{k+n}} CS^{k+n} \xhookrightarrow{C\alpha_n \cup \bar{\alpha_n}} CY_n \cup_{Y_n} Cf_n$$

where the last map is the map of pushouts induced by the maps on the diagrams defining them. Now since CY_n is contractible, we can collapse the factor of $CY_n \subset CY_n \cup_{Y_n} Cf_n$ to a point, which is the same as collapsing the base of Cf_n to a point, so it makes ΣX_n. In other words, there is a homotopy equivalence

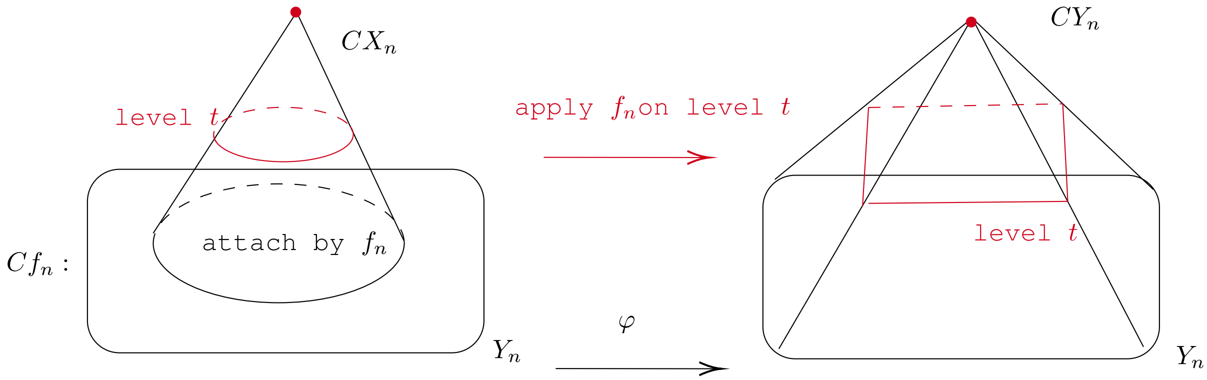
$$CY_n \cup_{Y_n} Cf_n \xrightarrow{\sim} \Sigma X_n.$$

From here we define

$$\beta_{n+1} \coloneqq S^{k+n+1} \xrightarrow{\beta'_{n+1}} CY_n \cup_{Y_n} Cf_n \xrightarrow{\sim} \Sigma X_n \xrightarrow{\xi_n} X_{n+1}.$$

Let $\beta \in \pi_k(X)$ be now the representative of β_{n+1} in the stable homotopy group. We claim that $f_*(\beta) = \pm\alpha$, which will then show the proof.

Indeed, consider the map $\varphi : Cf_n \to CY_n$ that "applies f_n to the cone on X_n". To be clear, this is the picture:



Now observe φ fits in a commutative diagram of the form:

$$\begin{tikzcd}
S^{k+n+1} \cong CS^{k+n} \cup_{S^{k+n}} CS^{k+n} \arrow[r, "C\alpha_n \cup \bar{\alpha}_n"] & CY_n \cup_{Y_n} Cf_n \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\cup\varphi"] & \Sigma X_n \arrow[r, "\xi_n"] \arrow[d, "\Sigma f_n"] & X_{n+1} \arrow[d, "f_{n+1}"] \\
& CY_n \cup_{Y_n} CY_n \arrow[r, "\sim"] & \Sigma Y_n \arrow[r, "\text{bond map}"'] & Y_{n+1}
\end{tikzcd}$$

where $\sim$ collapses the left side. Now going on the top line and then down along f_{n+1} is the element $\pm f_*(\beta_{n+1})$.

If we let $\sim$ collapse on the right side instead, the maps in the diagrams are still homotopic to the original ones (so commutativity holds homotopically). Now going along the top line, and down in $\mathrm{id}\cup\varphi$ and to Y_{n+1} represents $\pm\alpha$ (the resulting map is just suspension of α_n).

The sign ambiguity comes from a general theme that for a space S the choice of equivalence $CS/S \cong \Sigma S$ has two identifications that differs by a flip. ■

The labor of the previous lemma now allows us to prove.

Lemma 6.10 (Lemma 2.4.13 of [Mal26]). Let $f : X \to Y$ be a map of spectra. There are natural stable equivalences : $\Sigma F_f \xrightarrow{\cong} Cf$. Applying adjoints, this equivalently gives a stable equivalence $\tilde{\epsilon} : Ff \xrightarrow{\cong} \Omega C_f$.

Proof. Consider the model of S^1 as $I = [0,1]$ with $0 \sim 1$. We define the map

$$\epsilon_n : (\Sigma Ff)_n = S^1 \wedge (X_n \times_{Y_n} PY_n) \to Cf_n = Y_n \cup_{X_n} CX_n$$

as follows. $S^1 \wedge (X_n \times_{Y_n} PY_n)$ has a point-set model as $S^1 \wedge (X_n \times_{Y_n} PY_n)$, which is a quotient of $I \times X_n \times_{Y_n} PY_n$ by our previous discussion. With some possible redundancies, a point in $(\Sigma(X \times_Y PY))_n$ can be explicitly written as (t, x, γ) for $t \in [0,1], x \in X_n, \gamma \in PY_n$.

Now we define

$$\epsilon_n(t, x, \gamma) = \begin{cases} \gamma(2t), t \leq \frac{1}{2} \\ (x, 2-2t), t \geq \frac{1}{2} \end{cases} \quad \in Cf_n.$$

In other words, for $t \leq \frac{1}{2}$, $\epsilon_n(t, x, \gamma)$ is in the base $Y_n \subseteq Cf_n$ indicated by the path. For $t \geq \frac{1}{2}$, $\epsilon_n(t, x, \gamma)$ is in the cone of X_n. This is well-defined. For (x, γ) fixed, this should be thought of as a loop in Cf_n that traces the path γ in

Y_n and $I \times \{x\}$ in CX_n.

Warning / Reminder: If Cf_n denotes the unreduced cone, this is not a loop. Here (and prior) Cf_n refers to the reduced cone (i.e. attaching $I \wedge X_n$ with 0 as basepoint), so this is a loop.

Now consider the following diagram:

$$\begin{tikzcd}
\Sigma\Omega X \arrow[r, "\Sigma\Omega f"] \arrow[d, "\text{counit}"] & \Sigma\Omega Y \arrow[r] \arrow[d, "\text{counit}"] & \Sigma(X \times_Y PY) \arrow[r] \arrow[d, "\epsilon"'] & \Sigma X \arrow[r, "\Sigma f"] \arrow[d, "\text{flip}"'] & \Sigma Y \arrow[d, "\text{flip}"] \\
X \arrow[r, "f"'] & Y \arrow[r] & Y \cup_X CX \arrow[r] & \Sigma X \arrow[r, "\Sigma f"] & \Sigma Y
\end{tikzcd}$$

where the top row is (applying Σ to) the fiber Puppe sequence of f, the bottom row is the cofiber Puppe sequence of f. Recall the counit maps are equivalences, in Corollary 6.7. One can check this diagram is homotopy commutative, keeping in mind a concrete point-set model of the map $\Sigma\Omega Y_n = S^1 \wedge \Omega Y_n \to Y_n$ is given by sending $(t \in S^1, f : S^1 \to Y_n) \mapsto f(t)$.

Now Proposition 6.4 and Lemma 6.9 implies the top row is exact after applying π_k. Lemma 6.9 implies the bottom row is exact after applying π_k. The four vertical maps around ϵ are all isomorphisms on π_k, so the Five Lemma implies that ϵ is an isomorphism on π_k for all k and hence a stable equivalence. ■

Finally, we prove a key stability property that a sequence is a fiber sequence if and only if it is also a cofiber sequence.

Proposition 6.11 (Proposition 2.4.14 of [Mal26]). The data (X, Y, Z, f, g, h) describes a cofiber sequence if and only if it describes a fiber sequence.

Proof. Consider the two cofiber Puppe sequences of $f : X \to Y$ and $g : Y \to Z$

$$Y \longrightarrow Cf \longrightarrow \Sigma X \xrightarrow{\Sigma f} \Sigma Y$$

$$Y \xrightarrow{g} Z \longrightarrow Cg \longrightarrow \Sigma Y$$

Observe the following diagram is homotopy commutative:

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, "="'] & Cf = CX \cup_X Y \arrow[r, "\text{collapse Y}"] \arrow[d, "h \cup g"] & \Sigma X = CX \cup_X CX \arrow[r, "\Sigma f"] \arrow[d, "Cf \cup h"] & \Sigma Y \arrow[d, "="] \\
Y \arrow[r, "g"] & Z \arrow[r] & Cg = CY \cup_Y Z \arrow[r, "\text{collapse Z}"'] & \Sigma Y
\end{tikzcd} \quad (\dagger)$$

Clearly the left and right squares are homotopy commutative. Here we give an un-verified sketch on why the middle square is homotopy commutative. Indeed, on the level of spaces (replacing X, Y, Z with X_n, Y_n, Z_n's). The down-right direction is the map

$$\Phi : CX_n \cup_{X_n} Y_n \to CY_n \cup_{Y_n} Z_n, (t, x) \in CX_n \mapsto h(t, x) \in Z_n \text{ and } y \in Y_n \mapsto g(y) = (1, y).$$

The right-down direction is the map

$$\Psi : CX_n \cup_{X_n} Y_n \to CY_n \cup_{Y_n} Z_n, (t, x) \in CX_n \mapsto \begin{cases} h(2t, x), t \le 1/2 \\ (2 - 2t, f(x)), t \ge 1/2 \end{cases}, y \in Y_n \mapsto *.$$

There is a homotopy between Φ and Ψ by considering, for $1 \ge s \ge 0$,

$$\Phi^s(t, x) := \begin{cases} h(\frac{2}{(1+s)}t, x), t \le \frac{(s+1)}{2} \\ (2 - \frac{2}{(1+s)}(t), f(x)), t \ge \frac{(s+1)}{2} \end{cases} \quad \text{and } \Phi^s(y) = (2 - \frac{2}{1+s}, y).$$

Note that $\Phi^1 = \Phi$ and $\Phi^0 = \Psi$.

Now consider the maps

$$\Sigma(f \times h) : \Sigma X \to \Sigma(Y \times_Z F(I, Z)) \text{ and } \epsilon : \Sigma(Y \times_Z F(I, Z)) \simeq \Sigma F_g \to CY \cup_Y Z = Cg.$$

One can check that $\epsilon \circ \Sigma(f \times h) = h \cup C_f$, and Lemma 6.10 shows that ϵ is a stable equivalence.

Now we have

$$\begin{aligned}
(X, Y, Z, f, g, h) \text{ is a cofiber sequence} &\iff h \cup g \text{ is a stable equivalence} && \text{By Definition} \\
&\iff h \cup Cf \text{ is a stable equivalence} \\
&\iff \Sigma(f \times h) \text{ is a stable equivalence} \\
& && \epsilon \circ \Sigma(f \times h) = h \cup C_f,\ \epsilon \text{ is stable equivalence} \\
&\iff f \times h \text{ is a stable equivalence} && \text{Corollary 6.6} \\
&\iff (X, Y, Z, f, g, h) \text{ is a fiber sequence} && \text{By Definition}
\end{aligned}$$

The equivalence for $h \cup g$ and $h \cup Cf$ follows from the homotopy commutativity of the diagram in (†) and the fact that the two rows are exact. ■

6.3 Homotopy Pushouts Agrees with Homotopy Pullbacks

The last thing we want to show is that homotopy pushouts agree with homotopy pullbacks. Consider the square:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[d, "k"] \\
C \arrow[r, "g"'] & D
\end{tikzcd}$$

To do this, we lie on the following fact deferred as exercise in problem sessions. We can extend the square above to:

$$\begin{tikzcd}
Ff \arrow[r] \arrow[d] & A \arrow[r, "f"] \arrow[d, "h"'] & B \arrow[r] \arrow[d, "k"] & Cf \arrow[d] \\
Fg \arrow[r] & C \arrow[r, "g"'] & D \arrow[r] & Cg
\end{tikzcd}$$

Exercise 6.12. The square is a homotopy pushout if and only if the induced map $Cf \to Cg$ is an equivalence. The square is a homotopy pullback if and only if the induced map $Ff \to Fg$ is an equivalence.

Corollary 6.13. The square above is a homotopy pushout if and only if it is a homotopy pullback.

Proof. Suppose the square is a homotopy pushout, then the induced map $\phi : Cf \to Cg$ is an equivalence. By Lemma 6.10, we also have that

$$Ff \xrightarrow{\cong} \Omega Cf \xrightarrow{\Omega(\phi)} \Omega Cg \xleftarrow{\cong} Fg.$$

Hence the square is a homotopy pullback. The converse direction follows similarly. One needs to check that the induced maps are the ones specified in the exercise, but this can be done. ■

6.4 Corollaries of Stability Theorems

Here we record down some corollaries of the stability theorems.

Proposition 6.14. For spectra X, Y, the canonical map $\pi_k(X) \oplus \pi_k(Y) \to \pi_k(X \vee Y)$ is an isomorphism.

Remark 6.15. This is very false for spaces. For example $\pi_2(S^1 \vee S^2)$ is not a finitely generated abelian group, but $\pi_2(S^1) = 0, \pi_2(S^2) = \mathbb{Z}$.

Proof. Observe there is a cofiber sequence of the form

$$X \to X \vee Y \to Y,$$

because there is a level wise identification of the cofiber. Now applying Lemma 6.9 gives an exact sequence of the form

$$... \to \pi_k(X) \to \pi_k(X \vee Y) \to \pi_k(Y) \to \pi_{k-1}(X) \to ...$$

Now the map $r : X \vee Y \to Y$ is split since the natural inclusion map $i : Y \to X \vee Y$ has $r \circ i \simeq \mathrm{id}_Y$. The first map $\pi_k(X) \to \pi_k(X \vee Y)$ is also split. This means that we can decompose the long exact sequence into a split short exact sequence

$$0 \to \pi_k(X) \to \pi_k(X \vee Y) \to \pi_k(Y) \to 0,$$

and the proof follows. ■

Definition 6.16. For X, Y spectra the level-wise inclusion of $X_n \vee Y_n \to X_n \times Y_n$ gives a map $X \vee Y \to X \times Y$.

Proposition 6.17. The map $X \vee Y \to X \times Y$ is a stable equivalence.

Proof. We consider the map $\pi_k(X \vee Y) \to \pi_k(X \times Y)$. By the universal property of coproducts and products, we also get maps

$$\pi_k(X) \oplus \pi_k(Y) \to \pi_k(X \vee Y) \to \pi_k(X \times Y) \to \pi_k(X) \times \pi_k(Y).$$

Now one can check that π_k commutes with finite products, so the map $\pi_k(X \times Y) \to \pi_k(X) \times \pi_k(Y)$ is isomorphism. Proposition 6.14 shows the first map is an isomorphism. We can also check that the composition of all three maps above is an isomorphism. These conditions together implies the map $\pi_k(X \vee Y) \to \pi_k(X \times Y)$ is an isomorphism. ■

As a corollary, we also get by induction:

Corollary 6.18. For spectra $X_1, ..., X_n$, $X_1 \vee ... \vee X_n$ is stably equivalent to $X_1 \times ... \times X_n$.

Definition 6.19. Let X be a spectrum. A retract of X is a spectrum A such that there exists maps $i : A \to X, p : X \to A'$ such that $p \circ i : A \to A'$ is a stable equivalence.

Proposition 6.20. Extend the $p : X \to A'$ to $F_p \xrightarrow{j} X \xrightarrow{p} A'$. The map $j \vee i : F_p \vee A \to X$ is a stable equivalence. Equivalently, $X \simeq A \vee Ci$.

Proposition 6.21. Let W be a spectrum, then $\bullet \wedge W$ sends cofiber sequences of spectra to cofiber sequences of spectra. By stability, this also sends fiber sequences to fiber sequences. Dually, $F(-, W)$ sends fibers sequences of spectra to fiber sequences of spectra.

7 Week 7: Cellular and CW Spectra by Kiran Luecke*, Expanded by Mattie Ji

Note: I did not live-tex notes for this lecture. I am filling in this section by merging discussions from Kiran's lectures and the contents specified by the course syllabus designed by David Mehrle.

7.1 Cellular Spectra and Examples

In this lecture we will introduce and discuss **cellular and CW spectra**, which should be thought of as Cell / CW complexes with **cells of negative dimensions**. We should first clarify:

1. A **CW complex** X is built from X^0 (a discrete collection of apoints) inductively into a sequence $X^1, X^2, X^3, ...$ such that X^i is built from X^{i-1} by only attaching i-cells (i.e., the definition of a CW complex in [Hat02]). Technically, we also say X^0 is built from $X^{(-1)} = \emptyset$ by attaching 0-cells.
2. A **cellular complex** X is built from the process above except we allow attachments of cells of any (not necessarily uniform) dimensions at each stage. We still keep $X^{(-1)} = \emptyset$.
3. A **relative cellular complex** is built from the process of cellular complex above built we take $X^{(-1)} = A$.

Recall that for a based cell complex X, its reduced suspension ΣX has a canonical structure as a based cell complex with one $(k+1)$-cell for each k-cell of X, except for the based point.

Definition 7.1. A spectrum X is:

1. A **cellular spectrum** if each X_n is a based cell complex and $\xi_n : \Sigma X_n \to X_{n+1}$ is the inclusion of a subcomplex, where ΣX_n is given the structure described above.
2. A **CW spectrum** if each X_n is a based CW complex and $\xi_n : \Sigma X_n \to X_{n+1}$ is the inclusion of a subcomplex, where ΣX_n is given the structure described above.

For a cellular spectrum X and k an integer, a **stable k-cell** α of X is a $(k+n)$-cell in X_n (that is not a base point). This stable k-cell α can be identified with a $(k+n+1)$-cell α' in $\Sigma X_n \subseteq X_{n+1}$ and so on.

For instance, a 0-cell in X_1 other than the base point is a stable-(-1)-cell. A 2-cell in X_4 is a stable-(-2)-cell, etc.

Example 7.2. Let K be a based cell complex, then $\Sigma^\infty K$ has one stable k-cell for each k-cell of K (other than the base point when $k = 0$).

Example 7.3. Let G be an abelian group and consider HG the Eilenberg-MacLane spectrum. Recall we can build each $K(G, n)$ explicitly as a cell complex. Let $HG'_0 = G$. If we can HG'_{n+1} instead as the mapping cylinder of the bonding map $\Sigma K(G, n) \to K(G, n+1)$ (where $K(G, n+1)$ denotes the original one), we observe that the mapping cylinder is homotopy equivalent to $K(G, n+1)$, and the map $\Sigma K(G, n) \to K(G, n+1)$ is now a standard inclusion. This gives a cellular spectrum HG'. The natural map $HG \to HG'$ identifies the two. (One can also compute the homotopy groups of HG' explicitly).

Here we discuss an important class of examples of cellular spectra known as **Thom spectra**.

Notation: For an unbased space K, we write $F_q(K)$ or $\Sigma^{-d} K$ (where $d \geq 0$) as $\mathrm{sh}^d \Sigma^\infty K_+$. In other words, it is the suspension spectrum shifted to start at the d-th place. The terminology is important enough that we give the following name.

Definition 7.4. The **free spectrum on the level** n of an ubased space K is $F_nK \simeq \Sigma^n\Sigma^\infty(K_+)$. By Lemma 2.6.9 of F_nK, this spectrum has the property that for any spectrum X,

$$\text{Maps } F_nK \to X \cong \text{Un-based Maps } K \to X_n$$

The Thom spectrum is an important class of example of these free spectra.

Definition 7.5. Let B be an unbased finite dimensional cell or CW complex. Let $E - E'$ be a virtual vector bundle (formall difference of vector bundles) over B. Recall since B is paracompact, it is possible to rewrite any such virtual vector bundle into a formal difference $\gamma = F - \epsilon^q$ where F is a vector bundle and ϵ^q is the trivial bundle of rank q. We define the **Thom spectrum** of γ as

$$\mathrm{Th}(\gamma) = F_q(\mathrm{Th}(F)),$$

where $\mathrm{Th}(F)$ is the Thom space of F.

Clearly from definition we see that these Thom spectra are cellular or CW spectra (depending on B). Here we give another example that we also call Thom spectra.

Definition 7.6. We define the spectrum MO where $\mathrm{MO}_n = \mathrm{Th}(\gamma^n \to \mathrm{BO}(n))$, where γ^n is the universal classifying bundle on $\mathrm{BO}(n)$. Now recall that

$$\Sigma\,\mathrm{Th}(\gamma^n) \cong S^1 \wedge \mathrm{Th}(\gamma^n) \cong \mathrm{Th}(\epsilon^1 \oplus \gamma^n).$$

The bonding map here is given by

$$\Sigma\,\mathrm{Th}(\gamma^n) \cong \mathrm{Th}(\epsilon^1 \oplus \gamma^n) \xrightarrow{canonical} \mathrm{Th}(\gamma^{n+1}),$$

as $\epsilon^1 \oplus \gamma^n$ is a natural vector bundle over $\mathrm{BO}(n)$, we have a canonical map between the Thom spaces above induced by the map in the pullback:

$$\begin{tikzcd} \epsilon^1 \oplus \gamma^n \arrow[r] \arrow[d] & \gamma^{n+1} \arrow[d] \\ BO(n) \arrow[r, "\epsilon^1\oplus -"] & BO(n+1) \end{tikzcd}.$$

This is also an example of a cellular spectrum.

7.2 Attaching Cells for Spectra, Cofibrant Replacement, and Whitehead Theorem

Recall in spaces, we can build a CW complex out of pushout diagrams of the form:

$$\begin{tikzcd} S^n \arrow[r, "f"] \arrow[d, hook] & X^n \arrow[d] \\ D^{n+1} \arrow[r] & X^{n+1} \end{tikzcd} \qquad f \in \pi_n(X^n)$$

where X^n depends only on the class $f \in \pi_n(X^n)$.

Informally, we wish to think of cellular attachment for spectra as something of the following (homotopy) pushout in spectra:

$$\begin{tikzcd} \Sigma^\infty S^n = \mathbb{S}^n \arrow[r, "f"] \arrow[d] & X^n \arrow[d] \\ * \arrow[r] & X^{n+1} \end{tikzcd} \qquad f \in \pi_n^{\mathrm{st}}(X^n) \tag{$\dagger$}$$

where the outcome only depends on the stable homotopy class of f.

Remark 7.7. Here X^{n+1} becomes a cofiber of f. Note this is very much not the case in spectra. The first diagram, after stabilization (taking suspension Σ^∞) could be interpreted as an example of the second diagram under the adjunction

$$\Sigma^\infty : \mathrm{Spaces}_* \rightleftarrows \mathrm{Sp} : \Omega^\infty.$$

Note however that after taking suspensions, it is possible to identify two non-homotopy-equivalent spaces as the same spectra. For example, consider the Hopf fibration

$$\eta : S^3 \to S^2.$$

Stably, the map η is actually a 2-torsion in the stable homotopy groups of spheres, albeit $\pi_2(S^3) = \mathbb{Z}$. If we take

$$X = S^2 \cup_{2\eta} D^4 \text{ and } Y = S^2 \vee S^4.$$

X and Y are not homotopy equivalent as spaces, but after taking Σ^∞ the two spectra are equivalent.

Let us look at two examples.

Example 7.8. Consider the map $2 : \mathbb{S}^0 \to H\mathbb{Z}$ given by multiplication by 2 on π_0. We consider the (homotopy) pushout diagram:

$$\begin{tikzcd} \Sigma^\infty S^0 = \mathbb{S}^0 \arrow[r, "2"] \arrow[d] & X^0 = H\mathbb{Z} \arrow[d] \\ * \arrow[r] & X^1 \end{tikzcd}.$$

Since X^1 is the cofiber, we have a cofiber sequence

$$\mathbb{S}^0 \xrightarrow{2} H\mathbb{Z} \to X^1 \to \Sigma\mathbb{S}^0 = \mathbb{S}^1.$$

Using Lemma 6.9, we have a long exact sequence of homotopy groups. Now $\pi_k H\mathbb{Z} = 0$ for $k \neq 0$, so we have that

$$\pi_i(X^1) \cong \pi_{i-1}\mathbb{S}, \text{ for } i \geq 2.$$

Furthermore since $\mathbb{S}$ has no negative homotopy groups, we also have that

$$\pi_i(X^1) = 0 \text{ for } i < 0.$$

For the last two places, we have that

$$\pi_1(H\mathbb{Z}) = 0 \to \pi_1(X^1) \to \pi_0(\mathbb{S}) = \mathbb{Z} \xrightarrow{\cdot 2} \pi_0(H\mathbb{Z}) = \mathbb{Z} \to \pi_0(X^1) \to 0.$$

This gives

$$0 \to \pi_1(X^1) \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \pi_0(X^1) \to 0,$$

which implies that

$$\pi_1(X^1) = \ker(2 : \mathbb{Z} \to \mathbb{Z}) = 0 \text{ and } \pi_0(X^1) = \mathbb{Z}/2.$$

Example 7.9. Let Hopf : $\mathbb{S}^3 \to \mathbb{S}^2$ be the Hopf fibration. This stably is a map $\eta : \mathbb{S}^1 \to \mathbb{S}^0$ where

$$\mathbb{S}^1 = \Sigma^{-2}\Sigma^\infty S^3 \xrightarrow{\Sigma^{-2}\,\mathrm{Hopf}} \Sigma^{-2}\Sigma^\infty S^2 = \mathbb{S}^0,$$

which by functoriality is the map

$$\Sigma^{-2}\Sigma^\infty(S^3 \xrightarrow{\mathrm{Hopf}} S^2).$$

Since Σ^∞ is a left adjoint and Σ^{-2} commutes with taking the cofiber, so the cofiber of the diagram is $\Sigma^{-2}\Sigma^\infty(\mathbb{C}P^2)$.

For all intents and purposes, $(\dagger)$ is a very nice way to describe attachments in CW spectra and was the approach taken in lecture. Here we also describe what a more general formulation from relative cell spectrum.

Definition 7.10 (Proposition 2.6.11 of [Mal26])**.** Let A be a spectrum. A map $f : A \to X$ is a **relative cellular spectrum** if it is a countable composition

$$A = X^{-1} \to X^0 \to X^1 \to \ldots \to X$$

where each $X^{j-1} \to X^j$ is the map in the pushout diagram:

$$\begin{tikzcd} \bigvee_i F_{n_i}(S^{(k_i-1)+n_i})_+ \arrow[r, "\bigvee_i f_i"] \arrow[d, "\bigvee_i F_{n_i}(S^{(k_i-1)+n_i} \hookrightarrow D^{k_i+n_i})"'] & X^{j-1} \arrow[d] \\ \bigvee_i F_{n_i}(D^{k_i+n_i})_+ \arrow[r] & X^j \end{tikzcd}.$$

Note: A cellular spectrum is when $A = *$.

Recall that CW approximation of spaces gives a way to turn a space X into a CW complex QX (up to weak equivalence), which can be thought of as a cofibrant replacement. Here we discuss this cofibrant replacement procedure on spectra.

Theorem 7.11 (Theorem 2.6.12 of [Mal26])**.** Let $f : A \to X$ be a map of spectra, then there is a (functorial) factorization

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "i"'] & X \\ & B \arrow[u, "g"'] \end{tikzcd},$$

where $i : A \to B$ is a relative CW spectrum and $B \to X$ is a level equivalence.

Proof Sketch. The proof essentially reduces to the level of spaces and ensureing compatibility between them. The statement for spaces, which is covered somehwere in [Hat02], is that a map $A_i \to X_i$ of spaces can be factored into

$$A_i \to B_i \to X_i,$$

where $A_i \to B_i$ is a relative CW complex and $B_i \to X_i$ is a weak equivalence. ■

Corollary 7.12. Let X be a spectrum, then X is level equivalent to a CW spectrum QX, and stably equivalent to a CW Ω-spectrum QRX.

Remark 7.13. Note that Q is a left deformation on Sp with respect to stable equivalences.

Proof. Take $A = *$ in Theorem 7.11, then B is a CW spectrum and $B \to X$ is a level equivalence. If we apply Theorem 7.11 to RX (the fibrant replacement of X to an Ω-spectrum) and $A = *$, then $B = Q(RX)$ is level equivalent to RX. One can check $Q(RX)$ is still fibrant and hence Ω-spectrum. Indeed, if we buy the fact that Q and R are cofibrant and fibrant replacements. Then in general QRX is constructed out of

$$0 \underbrace{\to}_{\text{cofibration}} QRX \underbrace{\to}_{\text{acyclic fibration}} RX.$$

Now compose one step further

$$0 \to QRX \underbrace{\to}_{\text{acyclic fibration}} RX \underbrace{\to}_{\text{fibration}} 0,$$

where the map $RX \to 0$ is fibrant, so QRX is both fibrant and cofibrant. Now, level equivalence of Ω-spectra implies stable equivalence, which concludes the proof. ■

Finally, we also state a Whitehead theorem for spectra.

Definition 7.14. Recall $f, g : X \to Y$ maps of spectra are homotopy equivalent if f_n, g_n are homotopic via $h_n : X_n \wedge I_+ \to Y_n$ that are compatible with bonding maps. This is equivalent to the data of $h : X \wedge I_+ \to Y$ or $h^+ : X \to F(I_+, Y)$.

The map $f : X \to Y$ is a homotopy equivalence of spectra if there is $g : Y \to X$ such that $f \circ g$ is homotopic to id_X and $g \circ f$ is homotopic to id_Y.

Theorem 7.15 (Proposition 2.6.16 of [Mal26]). Let $f : X \to Y$ be a level equivalence of cellular spectra, then it is a homotopy equivalence.

Since level equivalence and stable equivalence are equivalent on Ω-spectra, we have:

Theorem 7.16 (Whitehead Theorem for Spectra). Let X, Y be cellular Ω-spectra. A map $f : X \to Y$ is a stable equivalence if and only if it is a homotopy equivalence.

7.3 Postnikov and Whitehead Towers for Spectra

Let us first recall the theory of Postnikov and Whitehead tower **for spaces**.

Intuition: In spaces, we chose the cells "$S^0, S^1, S^2, \ldots$" to get a sequence of **measurement devices** to detect something about spaces. More precisely, we get a sequence of functors

$$\pi_n : \mathrm{Spaces}_* \to \mathrm{Set}, X \mapsto \pi_0 \mathrm{Map}_*(S^n, X).$$

Definition 7.17. We define $\mathrm{Spaces}_{*,\geq n}$ (resp. $\mathrm{Spaces}_{*,\leq n}$) be the full subcategory of Space_* on spaces X such that $\pi_k(X) = 0$ for $k < n$ (resp. $k > n$).

Here, $\mathrm{Spaces}_{*,\geq n}$ is called $(n-1)$-connected spaces and $\mathrm{Spaces}_{*,\leq n}$ is called n-truncated spaces.

Here we recall a general fact.

Proposition 7.18. The inclusion functor $\mathrm{Spaces}_{*,\geq n} \hookrightarrow \mathrm{Spaces}_*$ preserves colimits, and the inclusion functor $\mathrm{Spaces}_{*,\leq n} \hookrightarrow \mathrm{Spaces}_*$ preserves limits. This gives a left adjunction, called the **Postnikov truncation**,

$$\tau_{\leq n} : \mathrm{Spaces}_* \to \mathrm{Spaces}_{*,\leq n}\,.$$

Definition 7.19. The Postnikov tower is

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & \tau_{\leq 2}X \arrow[d] \\
 & \tau_{\leq 1}X \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] & \tau_{\leq 0}X
\end{tikzcd}$$

where the map $X \to \tau_{\leq k}X$ is the unit of the adjunction, and the maps $\tau_{\leq k}X \to \tau_{\leq k-1}X$ are fibrations.

We can dually define the **Whitehead tower** as a tower

$$\begin{tikzcd}
\tau_{>0}X \arrow[r] & X \\
\tau_{>1}X \arrow[u] \arrow[ur] & \\
\tau_{>2}X \arrow[u] \arrow[uur] & \\
\vdots \arrow[u] &
\end{tikzcd}\,.$$

The story for spectra is nearly identical, except now we have $\pi_n : \mathrm{Sp} \to \mathrm{Ab}$ for all $n \in \mathbb{Z}$, as opposed to $n \in \mathbb{Z}_{\geq 0}$.

Definition 7.20. We define the following:

1. $\mathrm{Sp}_{\geq 0}$, called **connective spectra**, as the full subcategory of spectra X such that $\pi_n(X) = 0$ for $n < 0$.
2. $\mathrm{Sp}_{>0}$, called **connected spectra**, as the full subcategory of spectra X such that $\pi_n(X) = 0$ for $n < 1$.
3. $\mathrm{Sp}_{>n}$, called n**-connected spectra**, as the full subcategory of spectra X such that $\pi_k(X) = 0$ for $k \leq n$.
4. $\mathrm{Sp}_{\leq n}$, called n**-truncated spectra**, as the full subcategory of spectra X such that $\pi_k(X) = 0$ for $k > n$.
5. The **heart of spectra**, $\mathrm{Sp}^{\heartsuit} = \mathrm{Sp}_{\geq 0} \cap \mathrm{Sp}_{\leq 0} \cong \mathrm{Ab}$, which can be identified as the Eilenberg-MacLane spectra.

We say a spectrum X is **bounded below** if $\pi_n(X) = 0$ for all n sufficiently low. We say a spectrum X is **bounded above** if $\pi_n(X) = 0$ for all n sufficiently large.

Remark 7.21. For $s \geq \mathbb{Z}$, $\mathrm{Sp}_{\geq s} \cap \mathrm{Sp}_{\leq s} = \{\Sigma^s HG\} \cong \mathrm{Ab}$.

Similar to the case of Postnikov towers for spaces, we can consider one for spectra. Before we do this, we discuss the relative homotopy groups of spectra first.

Definition 7.22. Let $f : A \to X$ be a map of spectra with bonding maps α_n and ξ_n respectively. The k-th **relative stable homotopy group** with respect to f, written $\pi_k(X, A)$, is the colimit

$$... \to \pi_{k+n}(X_n, A_n) \xrightarrow{\text{suspend}} \pi_{k+n+1}(\Sigma X_n, \Sigma A_n) \xrightarrow{\xi_n} \pi_{k+n+1}(X_{n+1}, A_{n+1}).$$

Note that there is a natural identification $\pi_k(X, A) \cong \pi_k(Cf)$.

These relative homotopy groups give us to kill off elements in the homotopy of a spectrum, as follows.

Proposition 7.23. Let $f : A \to X$ be a map of spectrum and $\alpha \in \pi_k(X, A)$. There is a factorization $A \to A' \to X$ such that the map

$$f_i : \pi_i(X, A) \to \pi_i(X, A')$$

is an isomorphism for $i < k$ and f_k is surjective with $\alpha \in \ker(f_k)$.

Intuition: Let us recall how this is done in the level of spaces. Let $\alpha \in \pi_k(X, A)$, this is represented as a map $\alpha_n : (D^{k+n}, S^{(k-1)+n}) \to (X_n, A_n)$. The boundary

$$\partial\alpha_n \coloneqq \alpha_n|_{S^{(k-1)+n}} : S^{(k-1)+n} \to A_n$$

gives an element in $\pi_{(k-1)+n}(A_n)$. We can attach a $k + n$-cell to A_n, making A'_n, along the map $\partial\alpha_n$, this has the effect of making $\partial\alpha_n$ null-homotopic in A'_n. We note that here the map

$$\pi_i(X_n, A_n) \to \pi_i(X_n, A'_n)$$

is an isomorphism for $i < k$ (by cellular approximation) and surjective for $i = k$ with kernel containing α.

Proof Sketch of Proposition 7.23. Make A' by constructing A'_n as in the outline above, which becomes a stable cell attachment. ■

Definition 7.24. The process of Proposition 7.23 is referred to as "killing off the element $\alpha \in \pi_k(X, A)$ by attaching cells".

This allows us to construct the Postnikov tower as follows.

Definition 7.25. Let X be a spectrum and $n \in \mathbb{Z}$, we construct a map

$$f_n : X \to P_n X$$

where is $P_n X$ is obtained by attaching cells to kill of $\pi_k(*, X)$ for $k \geq n + 2$ with respect to the map $X \to *$. To make the process canonical, we actually attach all possible cells to kill off the homotopy groups at each level. Here, $X \to P_n X$ is the successive amount of maps produced as part of Proposition 7.23, and $P_n X$ is called the **n-th Postnikov stage of** X.

By definition, we have that

$$\pi_i(*, X) \to \pi_i(*, P_n X) \text{ is an isomorphism for } i < n + 1.$$

$$0 \to \pi_i(*, P_n X) \text{ is a surjection for } i \geq n + 1.$$

The map $X \to P_n X$ induces an isomorphism $\pi_k(X) \to \pi_k(P_n X)$ for $k \leq n$. Note also that $\pi_k(P_n X) = 0$ for $k > n$.

We can construct a canonical map $\alpha_{n+1} : P_{n+1}X \to P_nX$ such that $\alpha_{n+1} \circ f_{n+1} = f_n$. Indeed, every cell used to build $P_{n+1}X$ is also used to build P_nX (which kills off one layer more). This means each space of $P_{n+1}X$ is really a subcomplex of that of P_nX, which gives a map $P_{n+1}X \to P_nX$. Clearly this commutes with the f_n's. Furthermore, clearly by construction here the map

$$\alpha_{n+1} : P_{n+1}X \to P_nX \text{ is an isomorphism on } \pi_{\leq n}.$$

The data above describes a **Postnikov tower**:

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_2 \arrow[d, "\alpha_2"'] \\
 & P_1X \arrow[d, "\alpha_1"'] \\
X \arrow[uur, "f_2"] \arrow[ur, "f_1"] \arrow[r, "f_0"] \arrow[dr, "f_{-1}"'] & P_0X \arrow[d, "\alpha_0"'] \\
 & P_{-1}X \arrow[d] \\
 & \vdots
\end{tikzcd}$$

Remark 7.26. By Definition II.4.12 of [Rud98], the Postnikov stages P_nX of a spectrum X (and maps $X \to P_nX$) is characterized uniquely up to the properties that:

1. $\pi_i(P_nX) = 0$ for $i > n$.
2. The map $(f_i)_* : \pi_k(X) \to \pi_k(P_nX)$ is an isomorphism for $k \leq n$.

Let $F \to P_nX \xrightarrow{\alpha_n} P_{n-1}X$ be a fiber sequence with F being the fiber of α_n. Using Lemma 6.9, running the long exact sequence shows that F has homotopy groups concentrated at degree n. It follows that $F \simeq \Sigma^n H\pi_n(X)$. This gives a fiber Puppe sequence in

$$\Sigma^n H\pi_n(X) \to P_nX \to P_{n-1}X \to \Sigma^{n+1}H\pi_n(X), \text{ for all } n \in \mathbb{Z}.$$

This gives the following definition.

Definition 7.27. The map $P_{n-1}X \to \Sigma^{n+1}H\pi_n(X)$ is the n-th **k-invariant** of the Postnikov tower.

In particular, when X is connective, $P_{-1}X \simeq *$, running the fiber Puppe sequence above when $n = 0$ gives:

Proposition 7.28. Let X be a connective spectra, then $H\pi_0X \to P_0X$ is an equivalence.

Example 7.29. Take $X = \mathbb{S}$ to be the sphere spectrum. Let us look at the first stages of the Postnikov tower of $\mathbb{S}$. Indeed, for $k < 0$, $P_k\mathbb{S} \simeq *$. For $k = 0$, we have $P_0\mathbb{S} = H\pi_0(\mathbb{S}) = H\mathbb{Z}$. For $k = 1$, we know there is a fiber Puppe sequence

$$\Sigma^1 H\mathbb{Z}/2 \to P_1\mathbb{S} \to H\mathbb{Z} \xrightarrow{k} \Sigma^2 H\mathbb{Z}/2,$$

where $k : H\mathbb{Z} \to \Sigma^2 H\mathbb{Z}/2$ is the k-invariant map.

Morally, we can think of $P_1\mathbb{S}$ as "an extension of $\Sigma H\mathbb{Z}/2$ and $H\mathbb{Z}$".

Proposition 7.30. There are two maps $H\mathbb{Z} \to \Sigma^2 H\mathbb{Z}/2$ up to homotopy, and k is the non-zero one.

Similar to the Postnikov tower, we can also make a Whitehead tower (a tower of connective covers). There is a way to get it from the Postnikov tower as follows.

Definition 7.31. Let $X\langle n\rangle$ as the fiber of $f_n : X \to P_{n-1}X$ and write $g_n : X\langle n\rangle \to X$. This has the effect that $X\langle n\rangle$ is n-connective and g_n is an isomporphism on $\pi_{\geq n}$. The map $\beta_n : X\langle n\rangle \to X\langle n-1\rangle$ is given by the map of fiber sequences:

$$\begin{tikzcd}
X\langle n\rangle \arrow[r] \arrow[d, dashed, "\beta_n"'] & X \arrow[r, "f_n"] \arrow[d, "="] & P_nX \arrow[d, "\alpha_n"] \\
X\langle n-1\rangle \arrow[r] & X \arrow[r, "f_{n-1}"] & P_{n-1}X
\end{tikzcd}$$

This now gives a tower:

$$\begin{tikzcd}
\vdots \arrow[d] & \\
X\langle 1\rangle \arrow[d, "\beta_1"'] \arrow[dr, "g_1"] & \\
X\langle 0\rangle \arrow[r, "g_0"] \arrow[d, "\beta_0"'] & X \\
X\langle -1\rangle \arrow[ur, "g_{-1}"'] \arrow[d] & \\
\vdots &
\end{tikzcd}$$

Example 7.32. We say a spectrum X is **graded Eilenberg MacLane** if $X \simeq \bigvee_{n\in\mathbb{Z}} \Sigma^n H\pi_n X$. We also sometimes write this as $\bigoplus_{n\in\mathbb{Z}} \Sigma^n H\pi_n X$.

Let Y be a space, then $F(\Sigma^\infty Y, H\mathbb{Z})$ is graded Eilenberg-MacLane. Indeed, one can identify

$$F(\Sigma^\infty Y, H\mathbb{Z}) \cong \bigoplus_{n\in\mathbb{Z}} \Sigma^n H(H^{-n}(Y;\mathbb{Z})).$$

Here $H^n(Y;\mathbb{Z}) \coloneqq \pi_0(\mathrm{Map}(\Sigma^{-n}Y, H\mathbb{Z}))$.

7.4 Homology and Cohomology of Spectra, Hurewicz Theorem

Cohomology is easier to define than homology, as we will only define homology of a spectra in coefficient A, for an abelian group A. Let us first cover cohomology first.

Definition 7.33. Let E be a spectrum. For a spectrum X, the E-cohomology group of X is the graded abelian groupp

$$E^*X := \pi_0 \operatorname{Map}(X, \Sigma^* E) = \pi_0 \operatorname{Map}(\Sigma^{-*}X, E).$$

(Here Map means $F(-,-)$). Let $E = HA$, we write

$$HA^*X \text{ as } H^*(X;\mathbb{Z}).$$

Now we look at homology.

Definition 7.34. Let X be a spectrum, we define $H_k(X;A)$ as the colimit of

$$\ldots \to H_{k+n}(X_n,*;A) \xrightarrow{\sigma,\cong} H_{k+n+1}(\Sigma X_n,*;A) \xrightarrow{\xi_n} H_{k+n+1}(X_{n+1},*;A) \to \ldots$$

where σ is the suspension isomorphism.

Clearly from the definition, we see that:

Proposition 7.35. Let $\Sigma^n_+ K$ be the suspension spectrum of a space K, then $H_k(\Sigma^n_+ K;A) \cong H_k(K;A)$, since ξ_n is also an isomorphism.

In particular, homology is an example of a **stable invariant** as it produces the same value on the space and its suspension spectrum.

Corollary 7.36. $H_k(\mathbb{S};A) = \tilde{H}_k(S^0;A) = \begin{cases} A, k = 0 \\ 0, k \neq 0 \end{cases}$.

Example 7.37. **Note:** Let X be a spectrum and $\Omega^\infty X$ be its infinite loop space. Although we have that $\pi_n(X) \cong \pi_n(\Omega^\infty X)$ for all n. It is in general **not true** that

$$H_n(\Omega^\infty X;A) \cong H_n(X;A) \text{ or } H_n(X,*;A).$$

For example, let $X = \Sigma^{2n} H\mathbb{Q}$. Then $H_*(X)$ is concentrated at $* = 2n$, but the zeroth space of X is $K(\mathbb{Q},2n)$, whose homology is not concentrated in a single degree.

Finally, we discuss how to obtain a **Hurewicz map** in spectra.

Definition 7.38. We define the **Hurewicz map** $\pi_k(X) \to H_k(X;\mathbb{Z})$ by applying the Hurewicz map as a map of colimits:

$$\begin{tikzcd}
\ldots \arrow[r] & \pi_{k+n}(X_n) \arrow[r, "\sigma"] \arrow[d, "h_{k+n}"] & \pi_{k+n+1}(\Sigma X_n) \arrow[r, "\xi_n"] \arrow[d, "h_{k+n+1}"] & \pi_{k+n+1}(X_{n+1}) \arrow[r] \arrow[d, "h_{k+n+1}"] & \ldots \\
\ldots \arrow[r] & H_{k+n}(X_n,*;\mathbb{Z}) \arrow[r, "\sigma{,}\cong"] & H_{k+n+1}(\Sigma X_n,*;\mathbb{Z}) \arrow[r, "\xi_n"] & H_{k+n+1}(X_{n+1},*;\mathbb{Z}) \arrow[r] & \ldots
\end{tikzcd}.$$

The Hurewicz map between spaces commutes with the colimits and hence gives a well-defined map.

Remark 7.39. Once we have defined a smash product $\wedge$ on spectra, this is alternatively the map

$$\pi_k(X \wedge \mathbb{S}) \to \pi_k(X \wedge H\mathbb{Z})$$

induced by the unit map $\mathbb{S} \to H\mathbb{Z}$.

There is a Hurewicz theorem associated to spectra as well.

Theorem 7.40. Let X be a spectrum. If X is $(k-1)$-connected then the Hurewicz map $\pi_k(X) \to H_k(X)$ is an isomorphism.

As a corollary, we see that:

Corollary 7.41. Let X, Y be bounded below spectra. A map $f : X \to Y$ is a stable equivalence if and only if it induces an isomorphism on homology with $\mathbb{Z}$-coefficients.

Proof of Theorem 7.40. We can reduce this to the case where X is a cellular spectrum with only i-cells for $i \geq k$ (technically this uses some Propositions in this book). This implies that X_n is $(k+n-1)$-connected and hence the Hurewicz theorem of spaces implies that the map $\pi_{k+n}(X_n) \to H_{k+n}(X_n, *; \mathbb{Z})$ is an isomorphism. ΣX_{n-1} is also $(k+n-1)$-connected, so the Hurewicz map for it is also an isomorphism. Taking the colimit of isomorphisms gives an isomorphism, which yields the proof. ■

8 Week 8: The Stable Homotopy Category of Spectra by Emmanuel Sackeyfio

The purpose of this talk is to show $\mathrm{Ho}(\mathrm{Sp})$ is stable, or really that it is triangulated. The following is one of many definitions of $\mathrm{Ho}(\mathrm{Sp})$ one can give.

8.1 The Stable Homotopy Category

Definition 8.1. The **stable homotopy category** $\mathrm{Ho}(\mathrm{Sp})$ is the homotopy category of Sp. Concretely, this is a category $\mathcal{C}$ where:

1. Objects of $\mathcal{C}$ are the same as Sp.
2. Morphisms $X \to_{\mathcal{C}} Y$ are zig-zags of maps where the backward maps are stable equivalences. For example, it may be of the form:

$$X \to X_1 \xleftarrow{\sim} X_2 \to X_3 \xleftarrow{\sim} X_4 \to X_5 \xleftarrow{\sim} X_6 = Y.$$

Phrased in another term, $\mathrm{Ho}(\mathrm{Sp})$ is the localization of Sp at stable equivalences. It satisies a universal property that for any functor $F : \mathrm{Sp} \to \mathcal{D}$ that sends stable equivalences to isomorphisms in $\mathcal{D}$, then F factors uniquely through $\mathrm{Sp} \xrightarrow{\text{localization}} \mathrm{Ho}(\mathrm{Sp})$.

Remark 8.2. There are several equivalent definitions of this considered in this book:

- In terms of CW spectra.
- In terms of Ω-spectra.
- In terms of eventually defined maps.

The upshot is that the homotopy category of all 3 models here are equivalent as they satisfy the universal property described here.

8.2 Additive Categories

Let us recall the general structure of additive categories.

Definition 8.3. Let $\mathcal{C}$ be a category; then $\mathcal{C}$ is additive if we have the following properties:

1. $\mathcal{C}$ admits finite products and coproducts (including the empty product and empty coproduct).
2. $\mathcal{C}$ has a zero object 0. This allows us to define a zero morphism $X \xrightarrow{0} Y$ as the composition of unique maps $X \to 0 \to Y$.
3. For all objects $X, Y \in \mathcal{C}$ we have that the natural map

$$\varphi : X \sqcup Y \to X \times Y$$

defined essentially by the matrix $\begin{pmatrix} \mathrm{id}_X & 0 \\ 0 & \mathrm{id}_Y \end{pmatrix}$ with universal properties, is an isomorphism.

Note that for $f, g : X \to Y$ two morphisms, we can define $f + g$ as the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{(f,g)} Y \times Y \xrightarrow{\varphi^{-1}} Y \sqcup Y \xrightarrow{\text{fold}} Y,$$

where $\Delta : X \to X \times X$ is the diagonal map fold : $Y \sqcup Y \to Y$ is identity on each component.

4. For each pairs $X, Y \in \mathcal{C}$, we have a group structure on $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, ie. for $f : X \to Y$, there is $(-f) : X \to Y$ such that $f + (-f) = 0$.

Furthermore, an additive category $\mathcal{C}$ is **abelian** if:

1. For $f : X \to Y$, there exist $\ker(f)$ and $\mathrm{coker}(f)$. Here $\ker(f)$ is the pullback

$$\begin{tikzcd} \ker(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

and $\mathrm{coker}(f)$ is the pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{coker}(f) \end{tikzcd}$$

2. The natural map

$$\mathrm{coker}(\ker(f) \to X) \to \ker(Y \to \mathrm{coker}(f))$$

is an isomorphism.

Here are some examples.

Example 8.4. 1. Ab, the category of abelian groups, is abelian.

2. For any commutative ring R, Mod_R is abelian.

3. For a fixed field K, the category of even dimensional vector spaces over K is additive but not abelian (you can construct kernels that are odd-dimensional).

4. The category of Banach Spaces is not abelian because it fails the second property of being abelian above.

More categorifically, we note that: composition in an additive category is **bilinear**, in the following sense:

$$\mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C)$$

then we have that

$$g \circ (f + f') = g \circ f + g \circ f' \text{ and } (g + g') \circ f = g \circ f + g' \circ f.$$

In other words, $\mathcal{C}$ is "enriched over Ab".

Proposition 8.5. $\mathrm{Ho}(\mathrm{Sp})$ is additive.

Proof Sketch. We verify the conditions in varying degrees of details.

1. We want to show that $[X, Y]_S$, the collection of zig-zags from $X \to Y$, is an abelian group for any given X and Y. For any two maps f, g in $\mathrm{Ho}(\mathrm{Sp})$, we define addition as the composition

$$X \xrightarrow{\Delta} X \vee X \xrightarrow{(f,g)} Y \vee Y \to Y,$$

where the map $\Delta : X \to X \vee X$ is induced by the pinch map $S^1 \to S^1 \vee S^1$ (how to quotient a circle to a figure 8). Using an identification X' with $\Sigma X' \cong X$, we can view Δ as the composition:

$$X \simeq \Sigma X' = (S^1 \wedge X') \to (S^1 \vee S^1) \wedge X' \to (S^1 \wedge X') \vee (S^1 \wedge X') \simeq X \vee X.$$

For $f : X \to Y$, we can definie the inverse $-f$ by "switching the suspension coordinates", when identifying $\Sigma X' \simeq X$.

2. We have **products**, **coproducts**, and 0 that were explained before.
3. The natural map $X \vee Y \to X \times Y$ is an isomorphism, as a consequence of the stability result given.
4. Composition is bilinear.

■

Warning: $\mathrm{Ho}(\mathrm{Sp})$ is not an abelian category.

Remark 8.6. The full subcategory of Eilenberg MacLane spectra is, however, abelian. From this philosophy, it seems like spectra is a thickening of an abelian category.

8.3 Triangulated Categories

We noted that $\mathrm{Ho}(\mathrm{Sp})$ is not an abelian category, but it does have the following structure.

Definition 8.7. A **triangulated category** $\mathcal{C}$ is the following data:

- (a) $\mathcal{C}$ is an additive category.
- (b) There is a translation functor $\mathcal{C} \to \mathcal{C}$, oftne noted by $X \mapsto X[1]$, that is an equivalence of categories.
- (c) We have a collection of distinguished triangles

$$X \to Y \to Z \to X[1]$$

  We denote the collection as Δ,

satisfying the following properties:

- (T1):
  - For all $f : X \to Y$, there exists a distinguished triangle

$$X \xrightarrow{f} Y \to Z \to X[1].$$

  - Δ is stable under isomorphisms.
  - $X \xrightarrow{id} X \to 0 \to X[1]$ is a distinguished traingle.
- (T2): $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangble if and only if

$$Y \xhookrightarrow{g} Z \xhookrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

  is also a distinguished triangle (ie. the 3rd map is obtained by applying the shift functor).
- (T3): Given

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

  there exists a map $Z \to Z'$ commuting with the diagram.

- (T4): Given triangles
$$X \to Y \to Y/X \to X[1], Y \to Z \to Z/Y \to Y[1], X \to Z \to Z/X \to X[1],$$
we have another triangle
$$Y/X \to Z/X \to Z/Y \to Y/X[1],$$
making the following diagram commute
$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] & & Z \arrow[rr] \arrow[dr] & & Z/Y \arrow[rr] \arrow[dr] & & Y/X[1] \\
& Y \arrow[ur] \arrow[dr] & & Z/X \arrow[ur] \arrow[dr] & & Y[1] \arrow[ur] & \\
& & Y/X \arrow[rr] \arrow[ur] & & X[1] \arrow[ur] & &
\end{tikzcd}$$

Remark 8.8. The last axiom T4 seems very arbitrary, but it was originally developed to axiomatize derived categories.

Example 8.9. Let $\mathcal{A}$ be an abelian category and $C_\bullet(\mathcal{A})$ be the category of chain complexes on $\mathcal{A}$. This has a model structure under quasi-isomorphisms, and we define the derived category of $\mathcal{A}$ as
$$\mathcal{D}(A) := \mathrm{Ho}(C_\bullet(\mathcal{A})).$$
This has a natural shift $A_\bullet \to A_\bullet[1]$ by
$$\begin{tikzcd}
\ldots \arrow[r] & A_{-1} \arrow[r] & A_0 \arrow[r] \arrow[d] & A_1 \arrow[r] & \ldots \\
\ldots \arrow[r] & A_{-2} \arrow[r] & A_{-1} \arrow[r] & A_0 \arrow[r] & \ldots
\end{tikzcd}$$
A good way to remember this indexing convention is - if we suspend HA, what degree is its homotopy group concentrated in?

Proposition 8.10. The category $\mathrm{Ho}(\mathrm{Sp})$ is triangulated.

Proof. We saw earlier that $\mathrm{Ho}(\mathrm{Sp})$ is additive.

- In this case, our shift operator is $\mathrm{sh}^1 : \mathrm{Sp} \to \mathrm{Sp}$ (see Definition 5.26), which descends to an endofunctor on the homotopy category. Using a slight abuse of notion, we will write Σ as the shift operator as $\Sigma X_n \simeq X_{n+1}$.
- We specify the distinguished triangles to be fiber / cofiber sequences. Note here we have that
$$X \xrightarrow{f} Y \to \mathrm{cofib}(f) \to X[1]$$
is distinguished if and only if
$$\mathrm{fib}(g) \to Y \xrightarrow{g} Z \to \mathrm{fib}(g)[1]$$
is distinguished.

To verify the axioms:

- For (T1)(i), clearly this follows from the existence of cofiber sequences.
- (T1)(ii) follows from the stability theorems as before.
- For (T1)(iii), clearly the following is a cofiber sequence

$$X \xrightarrow{id} X \to \mathrm{cofib}(id) = 0 \to X[1].$$

- For (T2), this follows because cofiber and fiber sequences are the same.
- For (T3), given the diagram:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & \mathrm{cofib}(f) \arrow[r] & X[1] \arrow[d] \\ X' \arrow[r, "f'"] & Y' \arrow[r] & \mathrm{cofib}(f') \arrow[r] & X'[1] \end{tikzcd}$$

The informal reason why we have a map is that $\mathrm{cofib}(f)$ can be roughly thought about as Y/X and $\mathrm{cofib}(f')$ similarly as Y'/X'. The map here is the map $Y \to Y'$ that behaves well after quotients on both sides. This is not a proof, only intuition.

- For (T4), omitted. The high level reason why is there is a distinguished triangle

$$\mathrm{cof}(g) \to \mathrm{cof}(f) \to \mathrm{cof}(h) \to \mathrm{cof}(g)[1].$$

■

Remark 8.11. There is another way to show $\mathrm{Ho}(\mathrm{Sp})$ is triangulated by showing Sp (which is built differently and means something different) is a stable ∞-category. For those who knows about ∞-categories, being stable means:

- Sp is pointed and admits cofibers.
- The suspension functor $X \to \Sigma X$ given by the pushout out

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

  is an equivalence.

It is a general fact that for a stable ∞-category $\mathcal{C}$, $\mathrm{Ho}(\mathcal{C})$ is triangulated. The proof here may be easier if one is willing to do the work to define what an ∞-category means.

There is also a way to define Sp out of excisive functors, albeit we skip the details here. Roughly speaking, one can write

$$\mathrm{Sp} = \mathrm{Exc}_*(S_*^{fin}, S_*)$$

as **reduced, excisive, functors for** $F : S_*^{fin} \to S_*$. Here,

1. Reduced means $F(0) = 0'$ (send zero to zero).
2. Excisive means F sends pushouts to pullbacks.

Spectra can be viewed as the stabilization of spaces with respect to Σ, and we can take spaces and by excisive.

9 Week 9: Smash Product by Austin Bell

Today we will be talking about the smash products of spectra. We first give a philosophical outline of what will happen today. This talk is also a story about how the speaker approached to understanding the material.

9.1 (Symmetric) Monoidal Categories

The first thing we want to talk about is a series of categorical concepts related to the idea of monoids.

Definition 9.1. A monoid is a triplet $(M, *, e)$ where M is a set, $* : M \times M \to M$ is a binary operation, and e is a unit with respect to $*$.

Everything about a monoid here is based on strict equalities. However, when we move to the concept of **monoidal categories**, we really wish to generalize this.

Definition 9.2. A **monoidal category** is the data $(\mathcal{C}, \odot, \mathbb{1}, \alpha, L, R)$ where:

1. $\mathcal{C}$ is a category.
2. $- \odot - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor.
3. $\mathbb{1} \in \mathrm{obj}(\mathcal{C})$, thought of as the identity (see unitators below).
4. α, the **associator**, gives a way to establish the **associative property**. Here α is a natural isomorphism
$$\alpha : (- \odot -) \odot - \xrightarrow{\sim} - \odot (- \odot -).$$
(Note: Here we say this gives a way because α is a choice of associativity, and the structure may differ based on the choice of α).

   To be clear for a pair of objects (X, Y, Z), α specifies a compatible isomorphism
$$\alpha_{X,Y,Z} : (X \odot Y) \odot Z \to X \odot (Y \odot Z).$$
5. L and R are **unitors** associated to $\mathbb{1}$. They are natrual isomorphisms
$$L : \mathbb{1} \odot - \simeq \mathrm{id} \text{ and } R : - \odot \mathbb{1} \simeq \mathrm{id}\,.$$

We require these data to satisfy the following coherence conditions (commutative diagrams).

The **associator pentagon**:

$$\begin{tikzcd}
 & (X \odot Y) \odot (A \odot B) \arrow[dr] & \\
X \odot (Y \odot (A \odot B)) \arrow[ur, "\alpha_{X,Y,A\odot B}"] \arrow[d] & & ((X \odot Y) \odot A) \odot B \\
X \odot ((Y \odot A) \odot B) \arrow[rr] & & (X \odot (Y \odot A)) \odot B \arrow[u]
\end{tikzcd}$$

and the **unitor triangle**:

$$\begin{tikzcd}
X \odot (1 \odot Y) \arrow[rr, "\alpha_{X,1,Y}"] \arrow[dr, "\mathrm{id}_X \odot L_Y"'] & & (X \odot 1) \odot Y \arrow[dl, "R_X \odot \mathrm{id}_Y"] \\
 & X \odot Y &
\end{tikzcd}$$

There are significant consequences of requiring the associator pentagon and unitor triangle to hold.

Definition 9.3. We say:

1. A **formal expression** is any formal combination of $\mathrm{obj}(\mathcal{C}), \odot, \mathbb{1}$.
2. A **formal composite** is any composition of natural isomorphisms α, L, R, and combining them together with $\odot$.
3. A **formal diagram** is a graph whose vertices are formal expressions and edges are formal composites.

Theorem 9.4 (MacLane Coherence Theorem)**.** In a **monoidal category**, all formal diagrams commute.

Now that we know what a monoidal category is, we quickly give some terminologies that will be useful.

Definition 9.5. If we do not require unitors to have inverses, then the data $(\mathcal{C}, \odot, \mathbb{1}, \alpha, L, R)$ is a **lax monoidal category**. If we require the unitors and associators to be equalities (i.e., the isomorphisms are identities), then the data $(\mathcal{C}, \odot, \mathbb{1}, \alpha, L, R)$ is a **strict monoidal category**.

Remark 9.6. In fact, MacLane Coherence Theorem can be reformulated as the statement that "every monoidal category is monoidally equivalent to a strict monoidal category".

Example 9.7 (See Examples in Section 4 of [Mal26])**.** There are 27 examples listed in the book. We mention only some of them:

1. $(\mathrm{GrMod}_k, \oplus, 0)$.
2. $(\mathrm{GrMod}_k, \otimes, k[0], \alpha_{\text{canonical}})$.
3. $(\mathrm{GrMod}_k, \otimes, k[0], \alpha_{\text{Koszul}})$.

A monoidal category is to a monoid what a symmetric monoidal category is to a commutative monoid. Here we define what it means to be a symmetric monoidal category.

Definition 9.8. A braiding on a monoidal category $(C, \odot)$ (abbreviated here) is a family of natural isomorphisms

$$\gamma_{A,B} : A \odot B \cong B \odot A$$

such that the following diagram holds:

$$\begin{tikzcd} a \odot \mathbb{1} \arrow[r, "\gamma_{a,\mathbb{1}}"] \arrow[d, "R"'] & \mathbb{1} \odot a \arrow[dl, "L"] \\ a & \end{tikzcd}$$

If the braiding is the same as its own inverse, which is the same as saying the following commutes:

$$\begin{tikzcd} a \odot b \arrow[r, "\gamma_{a,b}"] \arrow[dr, "="'] & b \odot a \arrow[d, "\gamma_{b,a}"] \\ & a \odot b \end{tikzcd}$$

then we say $(\mathcal{C}, \odot, \gamma)$ is a **symmetric monoidal category**.

We have the following criterion for when a homotopy category of a symmetric monoidal homotopical category is symmetric monoidal.

Lemma 9.9 (Lemma 4.1.8 of [Mal26]). Let $\mathcal{C}$ be a symmetric monoidal category and homotopical, $A \subset \mathcal{C}$ be a full subcategory such that:

1. $\odot$ preserve weak equivalences between $\mathrm{obj}(A)$.
2. $X, Y \in \mathrm{obj}(A)$, then $X \odot Y \in \mathrm{obj}(A)$.
3. $\mathbb{1} \in \mathrm{obj}(A)$.
4. We have a functor $Q : \mathcal{C} \to \mathcal{C}$ which lands in A.
5. We have a natural weak equivalence $q : QX \to X$

Then the left derived tensor, which we define

$$X \odot^L Y = QX \odot QY,$$

makes $(\mathrm{Ho}\,\mathcal{C}, \odot^L, \mathbb{1})$ a symmmetric monoidal category.

Example 9.10. A particular example here would be a symmetric monoidal category $(\mathcal{C}, \odot, \mathbb{1})$ that is also homotopic with a class of weak equivalences $W \subseteq \mathcal{C}$, and a left deformation $Q : \mathcal{C} \to \mathcal{C}$ that lands in a full subcategory $A = \mathcal{C}_Q$ containing the image of Q. The condition in Lemma 9.9 states the necessary compatibility conditions for we to give $\mathrm{ho}(\mathcal{C})$ a symmetric monoidal structure.

Here is another example of applying this lemma.

Example 9.11. Note that $(\mathrm{Top}_*, \bigwedge, S^0)$ is symmetric monoidal. One can check this satisfies the previous lemma, and hence $(\mathrm{HoTop}_*, \bigwedge^L, S^0)$ is a symmetric monoidal category.

Remark 9.12. A great account / reference for MacLane's theorem is in [Dug].

9.2 Adam's Handcrafted Smash Product

Now we pivot our attention to look at establishing smash products for spectra. Before we talk about the more canonical smash product, we first mention a **handcrafted smash product**, by Adams.

Remark 9.13. In [Mal26], this is called the **handicrafted smash product**.

Definition 9.14. Let X, Y be spectra. We define the handcrafted smash product

$$X \wedge Y \text{ level-wise by } (X \wedge Y)_n = X_{p_n} \wedge Y_{q_n},$$

for any (p_n, q_n) satisfying:

1. $(p_0, q_0) = (0, 0)$.
2. Every stage, either p_n or q_n increments.
3. p_n, q_n increase without bound as $n \to \infty$.

Let x_p, y_q denote the bonding maps for X, Y. Our bonding maps are given by either

$$X_p \wedge X_q \wedge S^1 \xrightarrow{\mathrm{id} \wedge y_q} X_p \wedge Y_{q+1},$$

or

$$X_p \wedge Y_q \wedge S^1 \to X_p \wedge S^1 \wedge Y_q \xrightarrow{(-1)^q} X_p \wedge S^1 \wedge Y_q \xrightarrow{x_p \wedge \mathrm{id}} X_{p+1} \wedge Y_q.$$

Remark 9.15. Note that this requires a choice of the (p_n, q_n). It turns out that they are all stably equivalent, so the choice is independent, but the proof is very hard.

However, the handcrafted smash product does have some undesirable conditions.

Remark 9.16. Sp is actually not symmetric monoidal under Adams' "handcrafted smash product". It turns out one only has a stable equivalence between $\mathbb{S} \wedge X$ and X as opposed to an isomorphism in general.

Given this, one could ask if we make a proposed category of spectra with some smash product structures, satisfying nice conditions?

Definition 9.17 ([Lew91]). Let Sp be a propsoed category of spectra with bifunctor $\bigwedge : \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$. We say this is a convenient category of spectra if:

- (A1): Sp is symmetric monoidal with respect to $\bigwedge$.
- (A2): There are adjunctions $\Sigma^\infty : \mathrm{Top}_* \rightleftarrows \mathrm{Sp} : \Omega^\infty$.
- (A3): The sphere spectrum, defined as $\Sigma^\infty S^0$ is the unit.
- (A4): There is either a loop natural transformation

  $$(\Omega^\infty D) \wedge (\Omega^\infty E) \to \Omega^\infty (D \wedge E),$$

  or a suspension natural transformation

  $$(\Sigma^\infty X \wedge Y) \to \Sigma^\infty(X) \wedge \Sigma^\infty(Y)$$

  that commutes with the associator, unitor, commutator isomorphisms of Top_* and Sp.
- (A5): There is a natural weak equivalence

  $$\theta : \Omega^\infty \Sigma^\infty X \to \lim_n \Omega^n \Sigma^n X.$$

Theorem 9.18 ([Lew91]). There is no convenient category of spectra.

9.3 The Smash-Product as a Black Box

We still would like $\mathrm{Ho}(\mathrm{Sp})$ to be symmetric monoidal. But let us do a formal derivation:

Question 9.19. What happens if we do have a hypothetical symmetric monoidal category $(\widehat{\mathrm{Sp}}, \wedge)$, with the same constructions like suspension spectra, fiber, cofiber, etc., such that $\mathrm{ho}(\widehat{\mathrm{Sp}}) \simeq \mathrm{ho}(\mathrm{Sp})$, and $\mathrm{ho}(\widehat{\mathrm{Sp}})$ is symmetric monoidal. What are some consequences of properties it will have?

The hope is may be we can construct some operations that satisfy these consequences. Here is a number of consequences that should hold:

- $\Sigma^\infty K \wedge X \cong K \wedge X$.
- $\Sigma^\infty A \wedge \Sigma^\infty K \cong \Sigma^\infty (A \wedge K)$.
- Smash product of cellular spectra is cellular.
- Smash product preserves colimits in each variable, so it preserves wedge sums, homotopy pushouts, etc.
- This satisfies the condition in Lemma 9.9, so $(\mathrm{Ho}(\widehat{\mathrm{Sp}}), \bigwedge^L)$ is symmetric monoidal.

There is a nice addendum to Lemma 9.9 that is nice to conclude this lecture.

Definition 9.20. A symmetric monoidal category $(\mathcal{C}, \odot, \gamma)$ is **closed** if each $- \odot b$ has a right adjoint, denoted $(-)^b$.

This should be thought of as an analog of the tensor-hom adjunction.

Lemma 9.21. Let $\mathcal{C}$ be a closed symmetric monoidal category and satisfies the hypothesis in Lemma 9.9 and the additional requirements that, let B be a full subcategory of $\mathcal{C}$ as well:

- Hom preserves weak equivalences on $A^{op} \times B$.
- Let $X \in \mathrm{obj}(A), Y \in \mathrm{obj}(B)$, then $\mathrm{Hom}(X, Y) \in \mathrm{obj}(B)$.
- $R : \mathcal{C} \to \mathcal{C}$ lands in B.
- There is a natural weak equivalence $c : X \to RX$.

Then it follows that there is a natural closed symmetric monoidal structure on $(\mathrm{Ho}(\mathcal{C}), \odot^L)$. Furthermore, the specified right adjoints are the **right derived hom**,

$$\mathrm{RHom}(X, Y) := \mathrm{Hom}(QX, RY).$$

It follows from this lemma that supposedly $\mathrm{Ho}(\widehat{\mathrm{Sp}}, \bigwedge^L)$ is closed symmetric monoidal.

9.4 Ring and Module Spectra

Assuming we have defined this black box smash product, we will now discuss the theory of ring spectra and modules over ring spectra.

To do this, we first record some terminologies about monoids over symmetric monoidal categories.

Definition 9.22. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal algebra. A **monoid** (or **algebra**) in $\mathcal{C}$ is an object R with maps

$$\mu : R \otimes R \to R, \eta : \mathbb{1} \to R$$

called the **multiplication map** and the **unit map**, such that the following diagrams commute:

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu \otimes \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \otimes \mu"'] & R \otimes R \arrow[d, "\mu"] \\ R \otimes R \arrow[r, "\mu"'] & R \end{tikzcd}$$

which should be thought of as the associativity statement that $(ab)c = a(bc)$, and

$$\begin{tikzcd} \mathbb{1} \otimes R \arrow[r, "\eta\otimes \mathrm{id}_R"] \arrow[dr, "\text{unitor}"'] & R \otimes R \arrow[d, "\mu"] & R \otimes \mathbb{1} \arrow[l, "\mathrm{id}_R \otimes \eta"'] \arrow[dl, "\text{unitor}"] \\ & R & \end{tikzcd}$$

whch should be thought of as the statement $1(a) = a = (a)1$.

We additionally say R is **commutative** if the following diagram commutes:

$$\begin{tikzcd} R \otimes R \arrow[rr, "\gamma"] \arrow[dr, "\mu"'] & & R \otimes R \arrow[dl, "\mu"] \\ & R & \end{tikzcd}$$

Given any algebra in a symmetric monoidal category, we can consider modules over them.

Definition 9.23. Let R be a monoid in the symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1})$. A left module over R is an object M with a map

$$\alpha : R \otimes M \to M$$

thought of as the action map, such that the following diagrams commute:

$$\begin{tikzcd} R \otimes R \otimes M \arrow[r, "\mu\otimes \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \otimes \mu"'] & R \otimes M \arrow[d, "\alpha"] \\ R \otimes M \arrow[r, "\alpha"'] & M \end{tikzcd}$$

which should be thought of as action associativity $(ab)m = a(bm)$, and

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "\eta\otimes \mathbb{1}"] \arrow[dr, "\text{unitor}"'] & R \otimes M \arrow[d, "\alpha"] \\ & M \end{tikzcd}$$

which thought be thought of as the statement $1(m) = m$.

One can similarly define a **right module** over R.

Example 9.24. Let $\mathcal{C} = \mathrm{Ab}$, symmetric monoidal under tensor product. A monoid over $\mathcal{C}$ is the same as a ring R, and a left module M over R is the same as a left R-module. A commutative monoid is the same as a commutative ring.

Example 9.25. The unit of a symmetric monoidal category is a commutative monoid object. Every object is a module over the unit object.

This allows us to talk about the general theory of ring spectrum.

Definition 9.26. A ring spectrum R is a monoid object in $(\widehat{Sp}, \wedge, \mathbb{S})$. A spectrum X is an R-module if it is a module object over R. In particular, the sphere spectrum $\mathbb{S}$ is a commutative ring spectrum and every spectrum is a module over $\mathbb{S}$.

Now we record some properties about functors between (symmetric) monoidal categories.

Definition 9.27. Let $(\mathcal{C}, \otimes_{\mathcal{C}}, 1_{\mathcal{C}})$ and $(\mathcal{D}, \otimes_{\mathcal{D}}, 1_{\mathcal{D}})$ by symmetric monoidal categories. A **lax symmetric monoidal functor** is the data of a functor $F : \mathcal{C} \to \mathcal{D}$ with natural morphisms $i : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ and

$$m : F(X) \otimes_D F(Y) \to F(X \otimes_{\mathcal{C}} Y),$$

for all $X, Y \in \mathcal{C}$.

We say a lax symmetric monoidal functor is a **strong symmetric monoidal functor** if the maps m and i above are additionally isomorphisms.

Proposition 9.28. Let $F : \mathcal{C} \to \mathcal{D}$ be a lax symmetric monoidal functor, and R be a monoid (resp. commutative monoid) in $\mathcal{C}$, then $F(R)$ is a monoid (resp. commutative monoid) in $\mathcal{D}$. Furthermore, if M is a module over R, then $F(M)$ is a module over $F(R)$.

Idea. The proof is formal by diagram chasing. ■

Proposition 9.29. Let $F : \mathcal{C} \to \mathcal{D}$ be strong symmetric monoidal and $G : \mathcal{D} \to \mathcal{C}$ be a right adjoint (i.e., $F \dashv G$), then G is lax symmetric monoidal.

Example 9.30 (Unstable Examples)**.** The following are important examples of lax/symmetric monoidal functors for spaces to keep in mind:

1. The forgetful functor $(\mathrm{Top}, \times, *) \to (\mathrm{Set}, \times, *)$ is strong symmetric monoidal. This sends topological monoids to their underlying ordinayr monoids.
2. The adding a disjoint based point functor $(-)_+ : (\mathrm{Top}, \times, *) \to (\mathrm{Top}_*, \wedge, S^0)$ is strong symmetric monoidal. The forgetful functor $(\mathrm{Top}_*, \wedge, S^0) \to (\mathrm{Top}, \times, *)$ is a right adjoint and hence lax symmetric monoidal. In particular, $(-)_+$ sends an unbased monoid to a based monoid.
3. Homology is lax symmetric monoidal from $(\mathrm{Top}, \times, *) \to (\mathrm{GrAb}, \otimes \mathbb{Z}[0])$ or $(\mathrm{Ch}(\mathbb{Z}), \otimes, \mathbb{Z}[0]) \to (\mathrm{GrAb}, \otimes, \mathbb{Z}[0])$.

Example 9.31 (Stable Examples)**.** The following are important examples of lax/symmetric monoidal functors for spectra to keep in mind:

1. The based and unbased suspension spectrum functors are strong symmetric monoidal:

$$\Sigma^\infty : (\mathrm{Top}_*, \times, *) \to (\widehat{\mathrm{Sp}}, \wedge, \mathbb{S})$$

$$\Sigma^\infty_+ : (\mathrm{Top}, \times, *) \to (\widehat{\mathrm{Sp}}, \wedge, \mathbb{S}).$$

In particular, when G is a group or a monoid, $\Sigma^\infty_+ G$ is called the spherical group ring of G, denoted $\mathbb{S}[G]$.

2. The Eilenberg-MacLane spectrum functor is lax symmetric monoidal.

$$H(-) : (\mathrm{Ab}, \otimes, \mathbb{Z}) \to (\widehat{\mathrm{Sp}}, \wedge, \mathbb{S}).$$

3. The stable homotopy groups is lax symmetric monoidal from

$$\pi_* : (\mathrm{Ho}(\widehat{\mathrm{Sp}}), \wedge^L, \mathbb{S}) \to (\mathrm{GrAb}, \otimes, \mathbb{Z}[0]).$$

Corollary 9.32. Let R be a ring spectrum and M a module over R, then $\pi_*(R)$ is a graded ring and $\pi_*(M)$ is a graded module over $\pi_(R)$.

Finally, we end the discussion by mentioning some properties about dualizable spectra.

Definition 9.33. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal category. An object X is **dualizable** if there is another object X^*, called the dual, such that $X \otimes (-)$ is right adjoint to $X^* \otimes (-)$.

We state the following theorem without proof (since the smash product is still a blackbox).

Theorem 9.34. A spectrum X is dualizable if and only if X is a **finite spectrum**, meaning that X has finitely many stable cells.

10 Week 10: Symmetric and Orthogonal Spectra by Luke Barbarita

The outline of the talk is as follows:

1. Definitions and Two Approaches to Symmetric and Orthogonal Spectra.
2. Homotopy and Issues with Symmetric Spectra.
3. Finally Defining the Smash Product.

Recall: Last time, we noted that Sp is not the right category we wanted to work in, because it does not have a "good smash product", in the sense of [Lew91].

Definition 10.1. We refer to the objects in the category Sp we have been working with as **sequential spectra** and write now as $\mathrm{Sp}^{\mathbb{N}}$.

In this lecture, we will look at two other notions - symmetric and orthogonal spectra.

10.1 Symmetric and Orthogonal Spectra

Notation: We write Σ_n for the n-th symmetric group and $O(n)$ for the n-th orthogonal group.

Idea: We want spectra with Σ_n or $O(n)$ actions on the n-th level that mimic the actions on S^n.

More formally, we have that:

Definition 10.2. A **symmetric spectrum** X is a spectrum that has a basepoint preserving left action of Σ_n on X_n for all $n \geq 0$, such that

$\epsilon_{m,n}$ the composite of n bonding maps

is $\Sigma_m \times \Sigma_n$-equivariant. In other words, we have successive bonding maps

$$\epsilon_{m,n} : X_m \wedge S^n \to X_{m+1}S^{n-1} \to \dots \to X_{m+n},$$

there are actions of Σ_m on X_m and Σ_n on S^n, and $\Sigma_m \times \Sigma_n \subseteq \Sigma_{m+n}$ acts on X_{m+n}. We want this map to be $\Sigma_m \times \Sigma_n$-equivariant.

Definition 10.3. An **orthogonal spectra** X is a spectrum with a **continuous basepoint preserving lect action** of $O(n)$ such that ϵ_{m+n} is $O(m) \times O(n)$-equivariant.

Definition 10.4. A map of symmetric (resp. orthogonal) spectra is just a map between spectra that comnutes with Σ_n (resp. $O(n)$) for all n respectively.

From here we combine the objects and morphisms to define:

1. Sp^{Σ} - the category of symmetric spectra.
2. Sp^{O} - the category of orthogonal spectra.

Note that we can think of $\Sigma_n \subseteq O(n)$ by taking symmetric matrices. Thus every $O(n)$-action defines a Σ_n-action. This shows that:

Proposition 10.5. An orthogonal spectrum is a symmetric spectrum. A symmetric spectrum is a sequential spectrum. These are forgetful functors

$$U^O_{\mathbb{N}} : \mathrm{Sp}^O \xrightarrow{U^O_\Sigma} \mathrm{Sp}^\Sigma \xrightarrow{U^\Sigma_{\mathbb{N}}} \mathrm{Sp}^{\mathbb{N}} .$$

Example 10.6. The sphere spectrum $\mathbb{S}$ is an orthogonal spectrum, and hence also symmetric. This is done by considering the action on $O(n)$ on $\mathbb{R}^n$ and viewing S^n as the 1-point-compactification of $O(n)$.

Example 10.7. More generally, the suspension spectrum $\Sigma^\infty A$ is an orthogonal spectrum by considering $O(n)$ acting on $A \wedge S^n$ by ignoring A and only acting on S^n.

Remark 10.8. A model also exists for Eilenberg-MacLane spectra HA. For symmetric spectra, this is defined via

$$(HA)_n \coloneqq A \otimes \tilde{\mathbb{Z}}[S^n]$$

the space associated to the simplicial set of dimensionwise tensor product of A with the reduced free simplicial abelian group generated by the simplicial n-sphere. See Example 2.7 of [Sch]. A model of HA as an orthogonal spectrum is explained in Construction 5.3.8 of [Sch18].

Here we explain a second, maybe more streamlined, way to define them using **diagram spectra**.

Definition 10.9. A **based diagram** is a functor $X : I \to \mathrm{Top}_*$. The functor category Top_*^I is the category of all such functors.

Example 10.10. A spectrum is equivalently a based diagram from the sphere category $\mathcal{S}$ to Top_*. Here $\mathcal{S}$ has objects for each integer $n \geq 0$, and the morphisms are

$$\mathcal{S}(m,n) = \begin{cases} S^{n-m}, n \geq m \\ *, n < m \end{cases} .$$

As a result, we see that:

Proposition 10.11. $\mathrm{Top}_*^{\mathcal{S}} \cong \mathrm{Sp}^{\mathbb{N}}$.

We want to now discuss how to extend this perspective to symmetric and orthogonal spectra. The way to do this is to add **additional morphisms** into the sphere category to account for additional morphisms.

Definition 10.12 (Definition 6.1.6 of [Mal26])**.** $\mathcal{S}^\Sigma$ has the same object as $\mathcal{S}$, and the morphisms are

$$\mathcal{S}^\Sigma(m,n) = \begin{cases} (\Sigma_n)_+ \wedge_{\Sigma_{n-m}} S^{n-m}, n \geq m \\ *, n < m \end{cases} .$$

Essentially we get a wedge of spheres, one for each Σ_n, glued along the relations that $(\sigma\tau, x) \sim (\sigma, \tau x)$ where $\sigma \in \Sigma_n, \tau \in \Sigma_{n-m}, x \in S^{n+m}$.

There are some subtleties in writing down compositions here, which we elaborate as follows.

First we introduce some notations, write $[n] = \{1, 2, ..., n\}$. For any subset $A \subseteq [n]$, we write $S^A \subseteq S^n$ to denote the one-point compactification of $\mathbb{R}^A \subseteq \mathbb{R}^n$. This has the property (†) that for $A, B \subseteq [n]$ disjoint, $S^A \wedge S^B \cong S^{A \sqcup B}$. Furhtermore, for any injective map $\alpha : [m] \to [n]$, we have a canonical identification $S^{[n]-\alpha([m])} \cong S^{n-m}$.

Note that we can rewrite $\mathcal{S}^\Sigma(m, n)$ when $n \geq m$ as:

$$\mathcal{S}^\Sigma(m, n) = \bigvee_{\alpha:[m]\hookrightarrow[n], \text{ injections}} S^{[n]-\alpha([m])},$$

by permuting the positions of the subset $[m]$ in $[n]$. Now given $x \in \mathcal{S}^{[n]-\alpha([m])}$ for an injection α and $y \in \mathcal{S}^{[p]-\beta([n])}$ for an injection β, we wish to produce an element in their "composition" by defining a natural homeomorphism

$$S^{[p]-\beta([n])} \wedge S^{[n]-\alpha([m])} \xrightarrow{\cong} S^{[p]-\beta\circ\alpha([m])}.$$

Indeed, this follows by the decomposition of sets

$$[p] - \beta \circ \alpha([m]) = ([p] - \beta([n])) \sqcup \beta([n] - \alpha([m])) = ([p] - \beta([n])) \sqcup ([n] - \alpha([m])),$$

and using (†).

Theorem 10.13. $\mathrm{Top}_*^{\mathcal{S}^\Sigma} \simeq \mathrm{Sp}^\Sigma$.

We can do this similarly with orthogonal spectra.

Definition 10.14 (Definition 6.1.9 of [Mal26])**.** $\mathcal{S}^O$ has the same object as $\mathcal{S}$, and the morphisms are

$$\mathcal{S}^O(m, n) = \begin{cases} O(n)_+ \wedge_{O(n-m)} S^{n-m}, n \geq m \\ *, n < m \end{cases}.$$

The definition for its composition structure is also involved. We first note that $\mathcal{S}^O(m, n)$ can be equivalently described as follows.

Notation:

1. For $i : V \subseteq W$ an inclusion of inner product space, we get $W - i(V)$ denote the orthogonal complement of V in W, and S^{W-V} to be the one-point compactification of $W - i(V)$.
2. Let $O(m, n)$ denote **the space of linear isometries** $\mathbb{R}^m \to \mathbb{R}^n$.
3. For $n \geq m$, there is a vector bundle $E(m, n) \to O(m, n)$ whose fibers over each isometric linear embedding $i : \mathbb{R}^m \to \mathbb{R}^n$ is the orthogonal complement $\mathbb{R}^n - i(\mathbb{R}^m)$.

The **Thom space** of $E(m, n)$ can be realized as $\mathcal{S}^O(m, n)$.

For $m \leq n \leq p$, there is a product map

$$E(m, n) \times E(n, p) \to E(m, p)$$

given by composing embeddings $i : \mathbb{R}^m \to \mathbb{R}^n, j : \mathbb{R}^n \to \mathbb{R}^p$ to $j \circ i$ in the base space and identifying $y \in \mathbb{R}^p - j(\mathbb{R}^n)$ and $x \in \mathbb{R}^n - i(\mathbb{R}^m)$ to the point $y + j(x) \in \mathbb{R}^p - j \circ i(\mathbb{R}^m)$.

Taking the Thom-space on both sides gives a map

$$\mathcal{S}^O(m,n) \wedge \mathcal{S}^O(n,p) \to \mathcal{S}^O(m,p).$$

This defines the desired **composition map**.

Remark 10.15. There is also a coordinate free way to do this, using a larger category $\mathcal{J}$. Look at 6.1 of [Mal26]. This is used frequently in equivariant homotopy theory.

Theorem 10.16. $\mathrm{Top}_*^{\mathcal{S}^O} \simeq \mathrm{Sp}^O$.

Definition 10.17. The free symmetric and orthogonal spectra on spaces K are

$$F_n^\Sigma(K) \coloneqq K \wedge \mathcal{S}^\Sigma(n,-) \text{ and } F_n^O(K) \coloneqq K \wedge \mathcal{S}^O(n,-).$$

Remark 10.18. We can do compute colimits, limits, homotopy colimits on the underlying spectra, as $\Sigma_n/O(n)$ actions carry over wedge sums and pushouts.

For homotopy limits, we restrict to Ω-spectra and it still works out.

10.2 Homotopies and Issues with Sp^Σ

There are model structures on $\mathrm{Sp}, \mathrm{Sp}^\Sigma, \mathrm{Sp}^O$ defined level-wise (injective). From this, we get that

$$\mathrm{Ho}(\mathrm{Sp}^{\mathbb{N}}) \simeq \mathrm{Ho}(\mathrm{Sp}^\Sigma) \simeq \mathrm{Ho}(\mathrm{Sp}^O).$$

We also have that every diagram spectra has an underlying sequential spectra, and hence stable homotopy groups.

Definition 10.19. A π_*-isomorphism of diagram spectra is a map $X \to Y$ between diagram spectra that induces an isomorphism on $\pi_*(X) \cong \pi_*(Y)$.

Remark 10.20. In Sp^Σ, a stable equivalence is not the same as a π_*-isomorphism, but the two are equivalent in $\mathrm{Sp}^{\mathbb{N}}$. This means that the forgetful functor

$$\mathrm{Ho}\,\mathrm{Sp}^\Sigma \text{ and } \mathrm{Ho}\,\mathrm{Sp}^{\mathbb{N}}$$

is actually NOT an equivalence of categories. Another way to see this is to check that

$$\pi_0(F_1^\Sigma S^1) \not\cong \pi_0(F_1 S^1).$$

However, the two are also equivalent in Sp^O.

Question 10.21. How can we remedy this?

To remedy this, we can discuss a correct notion of stable equivalence.

Definition 10.22. A **stable equivalence of diagram spectra** is a map $f : X \to Y$ such that for any other diagram spectra Z that is also an Ω-spectrum, we have a bijection between

$$[Y, Z]_{\mathcal{C}} \to [X, Z]_{\mathcal{C}}, g : Y \to Z \mapsto g \circ f.$$

Definition 10.23. The **true homotopy groups** (for Sp^{Σ}) is the right derived functor

$$R\pi_*(X) \coloneqq \pi_*(RX),$$

where RX is a fibrant replacement of X.

10.3 Defining the Smash Product

We will now construct the smash product, which we will see is actually equivalent to the handcrafted smash product from before, but does give the properties we want on symmetric and orthogonal spectra.

Definition 10.24. For $X, Y \in \mathrm{Sp}^{\Sigma}$, we have the smash product $X \wedge Y$ is defined such that

$$(X \wedge Y)_n = \bigvee_{p+q=n} (\Sigma_{p+q})_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)/\sim .$$

Here $\sim$ is the relation identifying the images

$$\begin{tikzcd}
(\Sigma_{p+q+r})_+ \wedge X_p \wedge Y_q \wedge S^r \arrow[r] \arrow[d] & (\Sigma_{p+q+r})_+ \wedge X_p \wedge Y_{q+r} \\
(\Sigma_{p+q+r})_+ \wedge X_p \wedge S^r \wedge Y_q \arrow[r] & (\Sigma_{p+q+r})_+ \wedge X_{p+r} \wedge Y_q
\end{tikzcd}$$

We can do the same for Sp^{O}, replacing Σ_n by $O(n)$.

Example 10.25. $\mathbb{S} \wedge \mathbb{S} \cong \mathbb{S}$, in either symmetric or orthogonal spectra. Note that including our permutations is crucial, as otherwise we want get the orbit space $(S^n)_{\Sigma_n}$ as level n, which is contractible.

Example 10.26. $\Sigma^\infty A \wedge \Sigma^\infty B \cong \Sigma^\infty(A \wedge B)$.

Example 10.27. Let $X \in \mathrm{Sp}^{\Sigma}$, we have that $X \wedge \mathbb{S} \cong X$. More generally, we have that $X \wedge \Sigma^\infty K \cong X \wedge K$ (where $X \wedge K$ was previously defined as level-wise smash product).

Remark 10.28. In [Mal26], this is described as thinking of X_n as morally having "secret sphere coordinates".

Proposition 10.29. There are natural isomorphisms

1. $X \wedge \mathbb{S} \cong X$ (unital).
2. $X \wedge Y \cong Y \wedge X$ (braiding).

3. $(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z)$ (associator).

The golden theorem we have now is:

Theorem 10.30. $\mathrm{Sp}^\Sigma, \mathrm{Sp}^O$ are closed symmetric monoidal under $\wedge$.

11 Week 11: Ring Spectra by Melissa Wei

A brief outline of the lecture today is:

1. Definitions and Basic Properties of Ring Spectra.
2. Modules over Ring Spectra.
3. Finally, we will tie these into some examples and give a preview for next week.

Last couple weeks, we have been talking about the smash product, which gives a symmetric monoidal category $(\mathrm{Ho}\,\mathrm{Sp}, \wedge^L)$.

Question 11.1. Why do we want to have these symmetric monoidal structures?

One reason why this is desirable is that we can use them to talk about ring spectra. Throughout this talk, we will use $\widehat{\mathrm{Sp}}$ to denote the blackbox symmetric monoidal category of spectra. We can think about this as $\mathrm{Sp}^\Sigma, \mathrm{Sp}^O$, or (remark: more generally, some model category that is "Quillen-equivalent" to the underlying ∞-category of Sp).

11.1 Ring Spectra and Basic Properties

Let us recall some relevant concepts discussed.

Definition 11.2. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. A **monoid** R over $\mathcal{C}$ is an object R equipped with multiplication and unit maps

$$\mu : R \otimes R \to R \text{ and } \eta : I \to R$$

satisfying the conditions that

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu\otimes 1_R"] \arrow[d, "1_R\otimes\mu"'] & R \otimes R \arrow[d, "\mu_R"] \\ R \otimes R \arrow[r, "\mu_R"'] & R \end{tikzcd}$$

$$\begin{tikzcd} I \otimes R \arrow[r, "\eta\otimes 1_R"] \arrow[dr, "\cong"'] & R \otimes R \arrow[d] & R \otimes I \arrow[l, "1_R\otimes\eta"'] \arrow[dl, "\cong"] \\ & R & \end{tikzcd}$$

It is additionally **commutative** if

$$\begin{tikzcd} R \otimes R \arrow[r, "\cong"] \arrow[d, "\mu"'] & R \otimes R \arrow[dl, "\mu"] \\ R & \end{tikzcd}$$

Definition 11.3. A ring spectra is a monoid in $\widehat{\mathrm{Sp}}$, we call theese $\mathbb{E}_1$-ring spectra. A commutative ring spectra is a commutative monoid in $\widehat{\mathrm{Sp}}$, we call these $\mathbb{E}_\infty$-ring spectra.

There is a lot more data one can associate to ring spectra.

Definition 11.4. A symmetric ring spectrum R consists of:

1. A sequence of spaces $\{R_n\}$ for $n \geq 0$.

2. An action of Σ_n on each R_n for $n \geq 0$.
3. A $\Sigma_n \times \Sigma_m$-equivariant map $\eta_{n,m} : R_n \wedge R_m \to R_{n+m}$.
4. Maps $\iota_0 : S^0 \to R_0, \iota_1 : S^1 \to R_1$ satisfying the conditions:

$$\begin{tikzcd} R_n \wedge R_m \wedge R_p \arrow[r, "id \wedge \mu_{m,p}"] \arrow[d, "\mu_{n,m} \wedge id"'] & R_n \wedge R_{m+p} \arrow[d, "\mu_{n,m+p}"] \\ R_{n+m} \wedge R_p \arrow[r, "\mu_{n+m,p}"'] & R_{n+m+p} \end{tikzcd}$$

and the unit condition

$$R_n \cong R_n \wedge S^0 \xrightarrow{id \wedge \iota_0} R_n \wedge R_0 \xrightarrow{\mu_{n,0}} R_n.$$

$$R_n \cong S^0 \wedge R_n \xrightarrow{\iota_0 \wedge id} R_0 \wedge R_n \xrightarrow{\mu_{0,n}} R_n$$

for all $n \geq 0$, and the centrality condition:

$$\begin{tikzcd} R_n \wedge S^1 \arrow[r, "id \wedge \iota_1"] \arrow[d, "\tau"'] & R_n \wedge R^1 \arrow[r, "\mu_{n,1}"] & R_{n+1} \arrow[d, "\chi_{n,1}"] \\ S^1 \wedge R_n \arrow[r, "\iota_1 \wedge id"'] & R_1 \wedge R^n \arrow[r, "\mu_{1,n}"'] & R_{1+n} \end{tikzcd}$$

Here τ denotes the swap back, and $\chi_{n,m}$ putts the first n-pieces in $n+m$ pieces to be behind the m-pieces.

Remark 11.5. If you unwrap what a commutative monoid means over symmetric spectra, this is the definition.

Example 11.6. The sphere spectrum $\mathbb{S}$ is initial among commutative ring spectrum.

Example 11.7. The Eilenberg-MacLane spectrum

$$(\mathrm{Ab}, \otimes, \mathbb{Z}) \xrightarrow{H(-)} (\widehat{\mathrm{Sp}}, \wedge, \mathbb{S})$$

is lax symmetric monoidal (i.e. there are natural maps $HA \wedge HB \to H(A \otimes B)$). For a discrete ring R, we have that

$$HR \wedge HR \to H(R \otimes R) \to HR$$

which gives HR a ring spectrum structure.

Example 11.8. The suspension spectra functor

$$\Sigma^\infty_+ : (\mathrm{Top}, \times, *) \to (\widehat{\mathrm{Sp}}, \wedge, \mathbb{S})$$

sends topological groups and monoids to ring spectrum. In particular for a topological group G, $\Sigma^\infty G_+$ is a called a spherical group ring.

Example 11.9. Stable homotopy groups π_* are lax symmetric monoidal:

$$(\mathrm{Ho}\,\mathrm{Sp}, \Lambda^L, \mathbb{S}) \xrightarrow{\pi_*} (\mathrm{GrAb}, \otimes, \mathbb{Z}[0]),$$

note here the target is symmetric monoidal with a braiding given by

$$a \otimes b \mapsto (-1)^{|a||b|} b \otimes a.$$

This gives

$$\pi_*(X) \otimes \pi_*(Y) \to \pi_*(X \wedge Y), \mathbb{Z} \to \pi_0(\mathbb{S}).$$

If R is a (commutative) ring spectrum, then $\pi_*(R)$ is a (graded-commutative) graded ring.

Note that for a ring HA, $\pi_*(HA)$ is the ring A concentrated at degree 0.

Remark 11.10. For the ring $\mathbb{S}$, $\pi_*(\mathbb{S})$ is the stable homotopy groups of spheres. It is a non-trivial fact of **Nishida's nilpotence theorem** that $\pi_{\geq 1}(\mathbb{S})$ is nilpotent. This was later generalized by Devinatz-Hopkins-Smith into what is called the **nilpotence theorem**.

Example 11.11. Finally, we state one more example for KU, where one has

$$\pi_*(\mathrm{KU}) = \mathbb{Z}[\beta^{\pm 1}], |\beta| = 2.$$

Definition 11.12. The category of ring spectra has objects ring spectra R and morphisms $f : R \to S$ of ring spectra, which by definition of maps that commute in:

$$\begin{tikzcd} R \wedge R \arrow[r, "f \wedge f"] \arrow[d, "\mu_R"'] & S \wedge R \arrow[d, "\mu_S"] \\ R \arrow[r, "f"'] & S \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \arrow[r, "="'] \arrow[d, "\eta_R"] & \mathbb{S} \arrow[d, "\eta_S"'] \\ R \arrow[r, "f"'] & S \end{tikzcd}$$

This is called a ring homomorphism.

11.2 Module Spectra

Now we consider the theory of module spectra.

Definition 11.13. An left R-module over a ring spectrum R is a spectrum M along with an action map $R \otimes M \to M$ such that such that the following diagrams commute:

$$\begin{tikzcd} R \otimes R \otimes M \arrow[r, "\mu \otimes \mathrm{id}_R"] \arrow[d, "\mathrm{id}_R \otimes \mu"'] & R \otimes M \arrow[d, "\alpha"] \\ R \otimes M \arrow[r, "\alpha"'] & M \end{tikzcd}$$

which should be thought of as action associativity $(ab)m = a(bm)$, and

$$\begin{tikzcd} \mathbb{1} \otimes M \arrow[r, "\eta \otimes \mathbb{1}"] \arrow[dr, "\text{unitor}"'] & R \otimes M \arrow[d, "\alpha"] \\ & M \end{tikzcd}$$

which thought be thought of as the statement $1(m) = m$.

One can similarly define a **right module** over R. When R is commutative, a left R-module is the same as a right R-module, so we just call them R-modules.

Definition 11.14. The category of module spectra $\mathrm{Mod}(R)$ has:

1. Objects being R-modules.
2. Morphisms $f : M \to N$ that commutes with the R-action:

$$\begin{tikzcd} R \wedge M \arrow[r, "id \wedge f"] \arrow[d, "\alpha_M"'] & R \wedge N \arrow[d, "\alpha_N"] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

A stable equivalence of R-module is defined to be a stable equivalence on underlying spectra.

Remark 11.15. Any object is a module over the unit in a symmetric monoidal category, so any spectrum is a module over the sphere spectrum.

Proposition 11.16. When R is a commutative ring spectrum, there is a natural closed symmetric monoidal structure on the category of R-modules.

This is constructed as follows.

Definition 11.17. Let R be a commutative ring spectrum and consider R-modules M, N. We define the relative tensor product $M \otimes_R N$ as the coequalizer of

$$\begin{tikzcd} M \wedge R \wedge N \arrow[r, shift left, "\alpha_M \wedge id_N"] \arrow[r, shift right, "id_M \wedge \alpha_N"'] & M \wedge N \end{tikzcd}$$

We also define $F_R(M, N)$ as the equalizer of

$$\begin{tikzcd} F(M, N) \arrow[r, shift left] \arrow[r, shift right] & F(R \wedge M, N) \end{tikzcd}$$

Proposition 11.18 (Hovey, Shipley and Smith, 2.2.2)**.** Let $\mathcal{C}$ be a cocomplete symmetric monoidal category and R a commutative monoid in $\mathcal{C}$, then the map

$$R \otimes - : \mathcal{C} \to \mathcal{C}$$

preserves coequalizers.

As a corollary we see that:

Corollary 11.19. $\otimes_R$ gives the category of R-modules a symmetric monoidal structure with R as the unit.

Proof Idea. Let Q be a monoid, M a right Q-module, and N a left Q-module. We can consider the coequalizer

$$M \otimes Q \otimes N \rightrightarrows M \otimes N.$$

Suppose M is a (R, Q)-bimodule, then $M \otimes_Q N$ is a left R-module. This allows us to consider the composition

$$M \otimes R \xrightarrow{\tau} R \otimes M \xrightarrow{\alpha} M.$$

Thus, M is a (R, R)-bimodule, so $M \otimes_R N$ is a left R-module. We can check the other side as well, and that R is a unit for $\otimes_R$. The symmetric monoidal structure follows from the one originally. ■

Example 11.20. If we take $\mathcal{C}$ to be $\mathcal{S}_*^\Sigma$ and R to be $\mathbb{S}$. Then $\mathbb{S}$-modules are exactly the category of symmetric spectra.

Now we have a symmetric monoidal category of R-modules. We can define an R-algebra as follows.

Definition 11.21. An R**-algebra spectrum** is a monoid in the category of R-modules.

Example 11.22. The category of $\mathbb{S}$-algebra spectra is exctly the category of ring spectra.

The following identification is more non-trivial but is explained very well in the lecture notes of [Shi17].

Example 11.23. The category of HA-algebra spectrum corresponds to commutative, differential graded, A-modules.

Theorem 11.24 (Schwede-Shipley)**.** Let $\mathcal{C}$ be a simplicial, cofibrantly generated, proper stable category, with a compact generator P, then we have a chain of simplicial Quillen equivalence between

$$\mathcal{C} \text{ and } \operatorname{End}(P)\text{-modules.}$$

Remark 11.25 (Theorem 5.1.6 of Schwede-Shipley)**.** There is an identification between $\operatorname{Ch}(A) \simeq$ HA-modules, using the theorem above. Furthermore, for a discrete ring A, the homotopy category of HA-modules is equivalent to the derived category of A.

Remark 11.26. As a final remark, we note that next week we will go into the theory of operads. Roughly speaking, an operad $\mathcal{O}$ in a symmetric monoidal category $\mathcal{C}$ is a collection:

- $\{\mathcal{O}(n)\}_{n\geq 0}$ with Σ_n action for $n \geq 0$.
- The maps
$$\gamma : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \times ...\mathcal{O}(k_n) \to \mathcal{O}(k_1 + ... + k_n).$$

An algebra X over an operad $\mathcal{O}$ has the structure of action maps

$$\mathcal{O}(n) \otimes X^n \to X \text{ for each } n.$$

Example 11.27. An example would be to take the operad $\mathrm{Assoc}(n) = \Sigma_n$ and $\mathrm{Comm}(n) = *$. It turns out that:

- Algebras over Assoc are exactly monoids in $\mathcal{C}$.
- Algebras over Comm are exactly commutative monoids in $\mathcal{C}$.

Thus other operads give different notions of monoids. There are also things like $\mathbb{E}_\infty, A_\infty$-operads, which we will talk more next time.

Under this perspective, we in fact have a stable ∞-category of spectra Sp. Here Sp satisfies a universal property that evaluation on $\mathbb{S}$ gives

$$\mathrm{Fun}^L(\mathrm{Sp}, \mathcal{D}) \cong \mathcal{D}.$$

This is really the free stable presentable ∞-category on one generator.

11.3 Non-Example of Ring Spectra

In general for a ring spectrum R, there is a map

$$\mu : R \wedge R \to R.$$

By the Tensor-Hom adjunction, this corresponds to a map

$$R \to \mathrm{Hom}_{\mathrm{RingSp}}(R, R).$$

Proposition 11.28. $\mathbb{S}/2$ is not a ring spectrum.

Here we use an argument due to Tom Goodwillie in [Goo12].

Proof. Applying the discussion above for $R = \mathbb{S}/2$ and taking π_0, we have a map

$$\pi_0(\mathbb{S}/2) \to \pi_0 \mathrm{Hom}_{\mathrm{RingSp}}(\mathbb{S}/2, \mathbb{S}/2).$$

One can show that $\pi_0(\mathbb{S}/2) = \mathbb{Z}/2$ (this follows from a cofiber calculation) and $\pi_0(\mathrm{Hom}_{\mathrm{RingSp}}(\mathbb{S}/2, \mathbb{S}/2)) = \mathbb{Z}/4$ (this follows from a more involved Steenrod algebra calculation). Since the functor π_0 is meant to be lax symmetric monoidal, the map

$$\mathbb{Z}/2 \to \mathbb{Z}/4 \text{ is a ring map.}$$

On the other hand, no such ring maps exist. ∎

11.4 Problem Session Addendum (by Kiran Luecke)

In the problem session, Kiran gave an alternative proof on why $\mathbb{S}/2$ is not a ring spectrum in more details.

Let us first look at the homotopy groups of $\mathbb{S}/2$. Recall that the cofiber sequence

$$\mathbb{S} \xrightarrow{\cdot 2} \mathbb{S} \to \mathbb{S}/2$$

induces a long exact sequence of homotopy groups

$$\begin{tikzcd}
 & & \cdots \arrow[dll] \\
\pi_2(\mathbb{S}) \arrow[r, "\cdot 2"'] & \pi_2(\mathbb{S}) \arrow[r] & \pi_2(\mathbb{S}/2) \arrow[dll] \\
\pi_1(\mathbb{S}) \arrow[r, "\cdot 2"'] & \pi_1(\mathbb{S}) \arrow[r] & \pi_1(\mathbb{S}/2) \arrow[dll] \\
\pi_0(\mathbb{S}) \arrow[r, "\cdot 2"'] & \pi_0(\mathbb{S}) \arrow[r] & \pi_0(\mathbb{S}/2) \arrow[dll] \\
0 & &
\end{tikzcd}$$

We know that $\pi_1(\mathbb{S}) = \pi_2(\mathbb{S}) = \mathbb{Z}/2$ and $\pi_0(\mathbb{S}) = \mathbb{Z}$. This shows that

$$\pi_0(\mathbb{S}/2) = \mathbb{Z}/2 \text{ and } \pi_1(\mathbb{S}/2) = \mathbb{Z}/2$$

There are two possibilities for $\pi_2(\mathbb{S}/2)$. It is either $\mathbb{Z}/4$ or $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.

Proposition 11.29. $\pi_2(\mathbb{S}/2) = \mathbb{Z}/4$.

Corollary 11.30. This shows $\mathbb{S}/2$ is not a ring spectrum.

Proof. For an $\alpha \in \pi_*\mathbb{S}/2$. We claim $\mathbb{S}/2$ being a ring spectrum would imply α is a 2-torsion. Indeed, if it is the case then $\pi_*\mathbb{S}/2$ is a ring and in particular has a unit element $1 \in \pi_0\mathbb{S}/2 = \mathbb{Z}/2$. Now observe that

$$2 \cdot \alpha = 2 \cdot (1 \cdot \alpha) = (2 \cdot 1) \cdot \alpha = 0 \cdot \alpha = 0.$$

■

Proof of Proposition 11.29. Consider the multiplication by 2 map

$$\cdot 2 : \mathbb{S}/2 \to \mathbb{S}/2.$$

Claim: The map factors as the composite

$$\mathbb{S}/2 \xrightarrow{\text{connecting map in cofiber sequence}} \mathbb{S}^1 \xrightarrow{\eta} \mathbb{S} \xrightarrow{\text{canonical cofiber map}} \mathbb{S}/2,$$

where η generates $\pi_1(\mathbb{S}) = \mathbb{Z}/2$.

Proof of Claim. To see why this factorization occurs, consider the map of cofiber sequences

$$\begin{tikzcd}
\mathbb{S}/2 \arrow[r, "\cdot 2"] & \mathbb{S}/2 \\
\mathbb{S} \arrow[r, "\cdot 2"] \arrow[u] & \mathbb{S} \arrow[u] \\
\mathbb{S} \arrow[r, "\cdot 2"] \arrow[u, "\cdot 2"] & \mathbb{S} \arrow[u, "\cdot 2"']
\end{tikzcd}$$

Call the map $\mathbb{S} \xrightarrow{p} \mathbb{S}/2$ as p. $2 \cdot p$ is null homotopic, so 2 factors through the cofiber of p. This gives a factorization

$$\begin{tikzcd} \mathbb{S}/2 \arrow[r, "\cdot 2"] \arrow[d] & \mathbb{S}/2 \\ \mathrm{cofib}(p) \cong \mathbb{S}^1 \arrow[ur, dashed] & \end{tikzcd}$$

The dotted map $\mathbb{S}^1 \to \mathbb{S}/2$ is an element of $[\mathbb{S}^1, \mathbb{S}/2] = \pi_1(\mathbb{S}/2) = \mathbb{Z}/2$. So there are two options. Either it is $\mathbb{S}^1 \xrightarrow{\eta} \mathbb{S} \to \mathbb{S}/2$, which is what we want, or it is the zero map. We wish to now show this is not 0. It then suffices to show that

$$\cdot 2 : \mathbb{S}/2 \to \mathbb{S}/2 \text{ is not null.}$$

This is equivalent to showing that $(\mathbb{S}/2)/2$ is not equivalent to $\mathbb{S}/2 \oplus \Sigma\mathbb{S}/2$ (which is the cofiber of the zero map). Now, we note that

1. $(\mathbb{S}/2)/2 \simeq \mathbb{S}/2 \otimes \mathbb{S}/2$.
2. $\cdot 2 : \mathbb{S} \to \mathbb{S}$ can be realized as $\Sigma^{-1}\Sigma^\infty$ of the double cover $S^1 \to S^1$. The cofiber of spaces gives $S^1 \to S^1 \to \mathbb{R}P^2$.
3. This implies that $\mathbb{S}/2 = \Sigma^{-1}\Sigma^\infty \mathbb{R}P^2$.
4. This implies that
$$(\mathbb{S}/2)/2 = \mathbb{S}/2 \otimes \mathbb{S}/2 = \Sigma^{-2}\Sigma^\infty(\mathbb{R}P^2 \times \mathbb{R}P^2).$$

Now we use mod 2 cohomology. Here, we write $H^* := H(-; \mathbb{Z}/2)$. By Kunneth formula

$$H^*(\mathbb{R}P^2 \times \mathbb{R}P^2) = \mathbb{Z}/2[a, b]/(a^3, b^3).$$

This has degree given by generators as:

1. Degree 0: 1.
2. Degree 1: a, b.
3. Degree 2: a^2, ab, b^2.
4. Degree 3: a^2b, ab^2.
5. Degree 4: a^2b^2.

Now we use some technologies from the Steenrod squares.

Claim: There are stable operations called $\mathrm{Sq}^1, \mathrm{Sq}^2$ such that

$$\mathrm{Sq}^2(ab) = a^2b^2 \text{ and } \mathrm{Sq}^2(x) = 0 \text{ if } |x| < 2.$$

The operations are stable in the sense that they act on $\Sigma^{-2}\Sigma^\infty$ of the space as well. In particular, this means that Sq^2 acts non-trivialy on $H^*(\mathbb{S}/2/2)$.

On the other hand

$$H^*(\mathbb{S}/2 \oplus \Sigma\mathbb{S}/2) = H^*(\mathbb{S}/2) \oplus H^{*-1}(\mathbb{S}/2) = H^{*+1}(\mathbb{R}P^2) \oplus H^*(\mathbb{R}P^2).$$

But $H^*(\mathbb{R}P^2) = \mathbb{Z}/2[a]/(a^3)$ and $\mathrm{Sq}^2(1) = 0, \mathrm{Sq}^2(a) = 1$ as they have degree less than 2. However, $\mathrm{Sq}^2(a^2) = 0$ as $a^4 = 0$ in the ring. ■

We have shown the factorization exists. Now, on the level of π_2, this is the map

$$\pi_2\mathbb{S}/2 \to \pi_2(\mathbb{S}^1) = \pi_1(\mathbb{S}) = \mathbb{Z}/2 \xrightarrow{\eta} \pi_2(\mathbb{S}) \to \pi_2(\mathbb{S}/2).$$

Reading the long exact sequence tells us that $\pi_2(\mathbb{S}/2) \to \pi_2(\mathbb{S}^1) = \mathbb{Z}/2$ is surjective. We also know that $\eta : \pi_2(\mathbb{S}^1) \to \pi_2(\mathbb{S})$ is an isomorphism (on generator η^2 generates $\pi_2(\mathbb{S})$). Finally, the map $\pi_2(\mathbb{S}) \to \pi_2(\mathbb{S}/2)$ is an injection from the long exact sequence. Thus, we have

$$\pi_2\mathbb{S}/2 \xrightarrow{\text{surjection}} \mathbb{Z}/2 \xrightarrow{\cong} \mathbb{Z}/2 \xrightarrow{\text{inject}} \pi_2\mathbb{S}/2.$$

In particular this means $\cdot 2 : \mathbb{S}/2 \to \mathbb{S}/2$ is not zero on $\pi_2(\mathbb{S}/2)$ which proves the claim. ■

Remark 11.31. Alternatively, $\cdot 2 : \mathbb{S}/2 \to \mathbb{S}/2$ being not null would also imply $\mathbb{S}/2$ is not a ring spectrum (with some work), instead of calculating $\pi_2(\mathbb{S}/2)$. The proof presented here is more roundabout.

Remark 11.32. It is however true that $\mathbb{S}/8$ is a ring spectrum, by a theorem of Burklund [Bur22]. The paper has a more general theorem of when the cofiber of a self-map on a ring spectrum is also a ring spectrum.

12 Week 12: Operads by Samuel Coyle*, Expanded by Mattie Ji

Note: I did not live-tex notes for this lecture because the speaker (Samuel Coyle) presented the lecture with beamer slides. I have filled in the contents here according to the syllabus content David Mehrle specified.

12.1 Motivation for Operads

So far, we have seen the notion a (symmetric/orthogonal) ring spectrum and the notion of ring spectrum over the homotopy category of spectra. The **latter are special cases** of the more general definitions of (commutative) monoids over a symmetric monoidal category, which we recall below:

Definition 12.1. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. A **monoid** R over $\mathcal{C}$ is an object R equipped with multiplication and unit maps

$$\mu : R \otimes R \to R \text{ and } \eta : I \to R$$

satisfying the conditions that

$$\begin{tikzcd} R \otimes R \otimes R \arrow[r, "\mu \otimes 1_R"] \arrow[d, "1_R \otimes \mu"'] & R \otimes R \arrow[d, "\mu_R"] \\ R \otimes R \arrow[r, "\mu_R"'] & R \end{tikzcd}$$

$$\begin{tikzcd} I \otimes R \arrow[r, "\eta \otimes 1_R"] \arrow[dr, "\cong"'] & R \otimes R \arrow[d] & R \otimes I \arrow[l, "1_R \otimes \eta"'] \arrow[dl, "\cong"] \\ & R & \end{tikzcd}$$

It is additionally **commutative** if

$$\begin{tikzcd} R \otimes R \arrow[r, "\cong"] \arrow[d, "\mu"'] & R \otimes R \arrow[dl, "\mu"] \\ R & \end{tikzcd}$$

However, the two notions are somewhat different. In general, we could have spectra that become (commutative) monoids over the homotopy category $\mathrm{ho}(\mathrm{Sp}) \simeq \mathrm{ho}(\mathrm{Sp}^\Sigma) \simeq \mathrm{ho}(\mathrm{Sp}^O)$, but the spectra themselves are not literally (commutative) ring spectra in any choice of $\widehat{\mathrm{Sp}}$.

The unstable story is also analogous. Over the symmetric monoidal category $(\mathrm{Top}_*, \times, *)$, a (commutative) monoid must be strictly associative, unital (and commutative), but often times we may encounter interesting spaces that are only homotopically associative, unitial (and commutative).

Example 12.2. For a pointed space X, the usual based loop space ΩX is not strictly associative or unitial, but it is up to homotopy. The usual double based loop space $\Omega^2 X$ is, additionally, commutative up to homotopy.

Remark 12.3. There is actually a model for loop space ΩX that makes it strictly unital and strictly associative (although not strictly commutative in general). This is called the **Moore loop space**.

From the examples above, one may also hope for a way to distinguish between different relaxed notions of commutativity. Intuitively, $\Omega^2 X$ and $\Omega^3 X$ are both homotopically commutative, but they should be distinguished from each other. This motivates us to expand on the usual definition of (commutative) monoids to consider various types of more relaxed structures.

Operads are a way to rigorously define these structures, which we will explain in this lecture.

> **Remark 12.4.** Operads were first defined by Peter May in [May72]. According to [MI22], Boardman and Vogt [BV68] implicitly defined a notion earlier that turns out to be equivalent. According to folklore, operads were originally invented to distinguish certain structures on the complex cobordism spectrum MU apart from that of the Brown-Petersen spectrum BP. Both spectra will be later defined in Week 14.

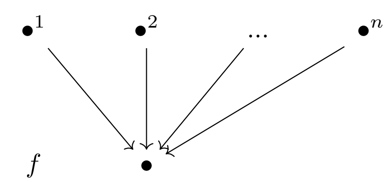
12.2 Definition of Operads

We will first discuss two types of operads: **symmetric and non-symmetric operads**. Often times in the literature, a symmetric operad is called an operad, so one often assumes an operad is symmetric unless specified otherwise. Here we follow a combination of [May72], [Bel17], [Sar], [MI22], [Bra] and [Cal21].

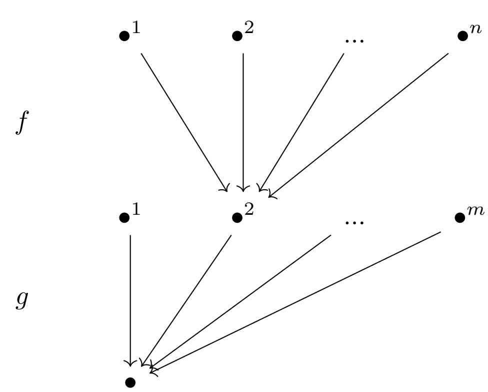
We first give an informal introduction to operads following [Bra] and then will present their formal definition (which may seem quite un-intuitive at first).

> **Definition 12.5.** An n-ary operation is an operation f that takes in n-inputs and produce 1 output.

We can visualize f as a strand of the form:



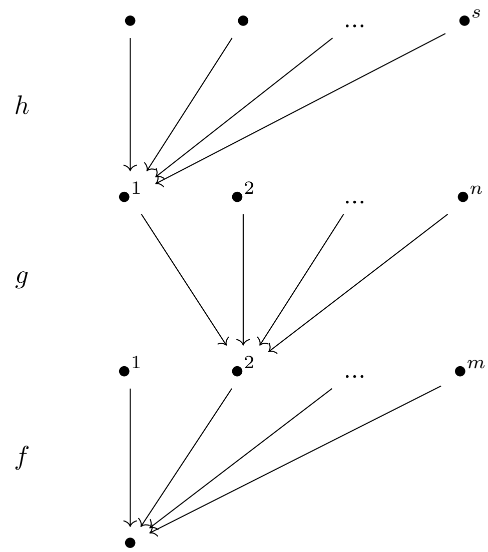
For two operations f and g, we can compose them as $g \circ_i f$ by placing the output of f at **the input of** g **labeled** i



This is for example, what $g \circ_2 f$ would look like. From this perspective, an operad is the data of how these operations compose. It should satisfy the following conditions:

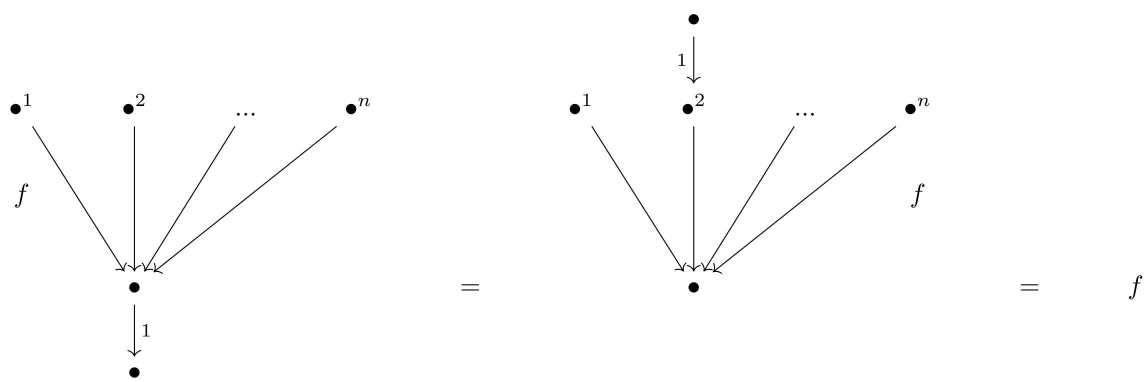
1. **(Associativity):** Composition should be associative in the following sense. For example, for operations f, g, h, if we compose $g \circ_1 h$ and then $f \circ_2 (g \circ_1 h)$, this should be the same as $(f \circ_2 g) \circ_1 h$. Pictorially, it means the

following two ways to build the same diagram should be the same.

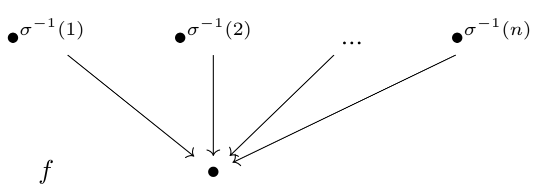


Similar associativity conditions should hold at other indices.

2. **(Identity):** There is a 1-ary operation called the identity, denoted 1, such that it is an identity with respect to composition. Pictorially:



(**Symmetry**): A symmetric operad has the additional action of a symmetric group Σ_n on the inputs of each n-ary operation that will permute the inputs. For $\sigma \in \Sigma_n$ and f an n-ary operation, we write σf as[1]:



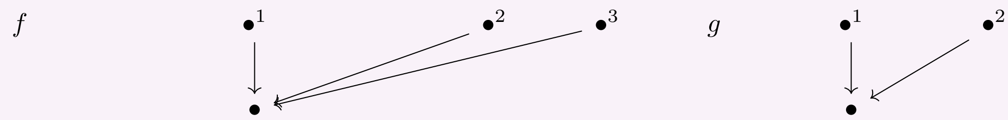
The composition should be well-behaved with the permutation in the sense that for $\sigma \in \Sigma_n$ and $\tau \in \Sigma_m$,

$$\sigma f \circ_k \tau g = (\sigma \circ_k \tau)(f \circ_k g),$$

where $\sigma \circ_k \tau$ permutes the entries of $f \circ_k g$ in the natural way.

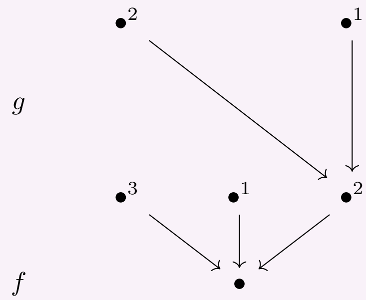
[1] As a convention, we actually apply σ^{-1}.

Example 12.6. Here is an example of the natural way from [Bra]. Consider

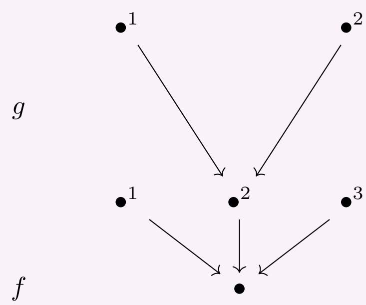


$\sigma = 1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$ $\qquad\qquad$ $\tau =$ flip

Then $\sigma f \circ_2 \tau g$ looks like:



This should be obtained from the following diagram



by the permutation that essentially "does τ but replace the action on 2 by τ".

Now we write down the more formal definition of operads with seek to encapsulate the discussions above.

Definition 12.7. Let $\mathcal{C}$ be a category. A (right) Σ_n-action on an object $c \in \mathcal{C}$ is a functor $F : \Sigma_n \to \mathcal{C}$, where we view Σ_n as a groupoid with 1 object, sending the unique object in Σ_n to c.

For $\sigma \in \Sigma_n$, we write a_σ to denote the right action by σ.

Example 12.8. For sets, vector spaces, algebras, spaces, etc., this coincides with the usual notion of a Σ_n-action.

Definition 12.9. Let $(\mathcal{C}, \otimes, I)$ be a symmetric monoidal category. A **non-symmetric operad** is a collection $\{\mathcal{O}(n)\}_{n \geq 0}$ of objects in $\mathcal{C}$ together with morphisms (called **composition rules**)

$$\gamma_{m;j_1,\ldots,j_m} : \mathcal{O}(m) \otimes \mathcal{O}(j_1) \otimes \ldots \otimes \mathcal{O}(j_m) \to \mathcal{O}(j) \coloneqq \mathcal{O}(j_1 + \ldots + j_m)$$

and a **unit map**

$$\eta : I \to \mathcal{O}(1),$$

satisfying the following axioms:

1. (**Associativity**): For any $m, j_1, ..., j_m$, and $k_1, ..., k_j$. We define

$$j \coloneqq j_1 + ... + j_m,$$

$$k \coloneqq k_1 + ... + k_j,$$

$$t_i \coloneqq j_1 + ... + j_{i-1} \text{ (with } t_1 = 0),$$

$$s_i = k_{t_i+1} + ... + k_{t_i+j_i}.$$

We require the following diagram to commute:

$$\begin{tikzcd}
& \mathcal{O}(j) \otimes \mathcal{O}(k_1) \otimes ...\mathcal{O}(k_j) \arrow[d, "\gamma_{j;k_1,\dots,k_j}"] \\
\mathcal{O}(m) \otimes \mathcal{O}(j_1)... \otimes \mathcal{O}(j_m) \otimes \mathcal{O}(k_1) \otimes ...\mathcal{O}(k_j) \arrow[ur, "\gamma_{m;j_1,\dots,j_m} \otimes id \otimes ... \otimes id"] \arrow[d, "c\text{, permutation}"'] & \mathcal{O}(k) \\
\mathcal{O}(m) \otimes (\bigotimes_{\alpha=1}^m \mathcal{O}(j_\alpha) \otimes \mathcal{O}(k_{t_\alpha+1}) \otimes ... \otimes \mathcal{O}(k_{t_{\alpha+1}})) \arrow[r, "id \otimes (\bigotimes_{\alpha=1}^m \gamma_{j_\alpha;k_{t_\alpha+1},\dots,k_{t_{\alpha+1}}})"'] & \mathcal{O}(m) \otimes \mathcal{O}(s_1)... \otimes \mathcal{O}(s_m) \arrow[u, "\gamma_{m;s_1,\dots,s_m}"']
\end{tikzcd}$$

where c is the natural permutation morphism of the factors given by the symmetric monoidal category.

2. (**Identity**): The unit map $\eta : 1 \to \mathcal{O}(1)$ is compatible with the compositions as follows. The composition

$$I \otimes \mathcal{O}(m) \xrightarrow{\eta \otimes id} \mathcal{O}(1) \otimes \mathcal{O}(m) \xrightarrow{\gamma_{1;m}} \mathcal{O}(m)$$

is the **unit isomorphism**, and the composition

$$\mathcal{O}(m) \otimes I^{\otimes m} \xrightarrow{id \otimes \eta^{\otimes m}} \mathcal{O}(m) \otimes \mathcal{O}(1)^{\otimes m} \xrightarrow{\gamma_{m;1,\dots,1}} \mathcal{O}(m)$$

is the **iterated unit isomorphism**.

A **symmetric operad** is a non-symmetric operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 0}$ equipped with the right action of Σ_n on each $\mathcal{O}(n)$, $n \geq 0$ such that:

1. The composition map $\gamma_{m;j_1,\dots,j_m}$

$$\mathcal{O}(m) \otimes \mathcal{O}(j_1) \otimes ...\mathcal{O}(j_m) \to \mathcal{O}(j)$$

is $(\Sigma_{j_1} \times ... \times \Sigma_{j_m})$-equivariant with respect to the $(\Sigma_{j_1} \times ... \times \Sigma_{j_m})$ action on the left side and the restriction of Σ_j to the subgroup $(\Sigma_{j_1} \times ... \times \Sigma_{j_m})$ on the right side.

2. For any $m, j_1, ..., j_m$ and $\sigma \in \Sigma_m$, we have that

$$\gamma_{m;j_1,\dots,j_m} \otimes (a_\sigma \otimes id_{\mathcal{O}(j_1)} \otimes ... \otimes id_{\mathcal{O}(j_m)}) = a_{\sigma_{j_1,\dots,j_m}} \circ \gamma_{m;j_{\sigma^{-1}(1)},\dots,j_{\sigma^{-1}(m)}} \circ (id_{\mathcal{O}(m)} \otimes c_\sigma).$$

Here $\sigma_{j_1,\dots,j_m}$ is the block permutation in Σ_j corresponding to σ under the inclusion map $\Sigma_{j_1} \times ... \times \Sigma_{j_m} \to \Sigma_j$, and c_σ is the permutation given in the symmetric monoidal category corresponding to σ.

Remark 12.10. Some literature [May72] requires $\mathcal{O}(0) = I$. This corresponds to the notion of unital operads.

Remark 12.11. If $(\mathcal{C}, \otimes, I)$ admits a forgetful functor to Set, we can think of elements of $\mathcal{O}(n)$, after forgetting to set, as n-ary operations. This aligns with the informal picture we illustrated earlier.

We can also discuss what morphisms between operads mean.

Definition 12.12. A morphism of non-symmetric operads $\phi : \mathcal{O} \to \mathcal{P}$ is a collection of morphisms $\phi_n : \mathcal{O}(n) \to \mathcal{P}(n)$ that respects the composition and unit maps. More specifically, the following diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(m) \otimes \mathcal{O}(j_1) \otimes \ldots \otimes \mathcal{O}(j_m) \arrow[r, "\gamma^{\mathcal{O}}_{m;j_1,\ldots,j_m}"] \arrow[d, "\phi_m \otimes \phi_{j_1} \otimes \ldots \otimes \phi_{j_m}"'] & \mathcal{O}(j) \arrow[d, "\phi_j"'] \\
\mathcal{P}(m) \otimes \mathcal{P}(j_1) \otimes \ldots \otimes \mathcal{P}(j_m) \arrow[r, "\gamma^{\mathcal{P}}_{m;j_1,\ldots,j_m}"'] & \mathcal{P}(j)
\end{tikzcd}$$

$$\begin{tikzcd}
 & I \arrow[dl, "\eta^{\mathcal{O}}"'] \arrow[dr, "\eta^{\mathcal{P}}"] & \\
\mathcal{O}(1) \arrow[rr, "\phi_1"'] & & \mathcal{P}(1)
\end{tikzcd}$$

This is equivalent to specifying maps $\mathcal{O}(n) \otimes X^{\otimes n} \to X$ satisfying relevant compatibility conditions.

A **morphism of symmetric operads** is a morphism of non-symmetric operads $\phi : \mathcal{O} \to \mathcal{P}$ such that each ϕ_n is Σ_n-equivariant.

12.3 Examples of Operads

The following is a central example of operads known as the **endomorphism operad**.

Definition 12.13. Let $\mathcal{C}$ be a **closed symmetric monoidal category** (e.g., spaces, sets, vector spaces) and X an object in $\mathcal{C}$. The **endomorphism operad** over $\mathcal{C}$ is a (symmetric) operad End_X given by

$$\mathrm{End}_X(n) := \mathrm{Map}_{\mathcal{C}}(X^{\otimes n}, X),$$

where $\mathrm{Map}_{\mathcal{C}}$ denotes the internal hom that exists for a closed symmetric monoidal category (it is the adjoint to the tensor product). In particular, there is a counit map called evaluating

$$\mathrm{ev} : \mathrm{Map}_{\mathcal{C}}(X, Y) \otimes X \to Y$$

1. The **unit map** $I \to \mathrm{End}_X(1) = \mathrm{Map}_{\mathcal{C}}(X, X)$ specifies the identity map.
2. The **composition map**

$$\gamma_{m;j_1,\ldots,j_m} : \mathrm{Map}_{\mathcal{C}}(X^{\otimes m}, X) \otimes \mathrm{Map}_{\mathcal{C}}(X^{\otimes j_1}, X) \ldots \mathrm{Map}_{\mathcal{C}}(X^{\otimes j_m}, X) \to \mathrm{Map}_{\mathcal{C}}(X^{\otimes j}, X)$$

   is given as follows. By adjunction it is enough to specify a map

$$\mathrm{Map}_{\mathcal{C}}(X^{\otimes m}, X) \otimes \mathrm{Map}_{\mathcal{C}}(X^{\otimes j_1}, X) \ldots \mathrm{Map}_{\mathcal{C}}(X^{\otimes j_m}, X) \otimes X^{\otimes j} \to X.$$

   which we can rewrite to

$$\mathrm{Map}_{\mathcal{C}}(X^{\otimes m}, X) \otimes (\mathrm{Map}_{\mathcal{C}}(X^{\otimes j_1}, X) \otimes X^{j_1}) \ldots (\mathrm{Map}_{\mathcal{C}}(X^{\otimes j_m}, X) \otimes X^{\otimes j_m}) \to X.$$

For each j_i, we have an evaluation map

$$\mathrm{ev}_{j_i} : \mathrm{Map}_{\mathcal{C}}(X^{\otimes j_i}, X) \otimes X^{j_i} \to X.$$

Collecting them over gives a map

$$\mathrm{id} \otimes \mathrm{ev}_{j_1} \otimes ... \mathrm{ev}_{j_m},$$

which lands in

$$\mathrm{Map}_{\mathcal{C}}(X^{\otimes m}, X) \otimes X^{\otimes m}.$$

We then compose this by ev_m to get a map into X.

This has the structure of a symmetric operad by letting Σ_n act on $X^{\otimes n}$ (for each $\mathrm{End}_X(n)$), according to the permutation maps given by the category $\mathcal{C}$.

Example 12.14. When $\mathcal{C} = \mathrm{Set}$, the endomorphism operad gives

$$\mathrm{End}_X(n) = \mathrm{Hom}_{Set}(X^n, X).$$

For $f : X^m \to X, g_i : X^{j_i} \to X$, the composition map is just

$$\gamma_{m;j_1,...,j_m}(f, g_1, ..., g_n) := f \circ (g_1 \times ... \times g_n).$$

The definition of endomorphism operad allows us to define the following.

Definition 12.15. Let $\mathcal{C}$ be a closed symmetric monoidal category and $\mathcal{O}$ an operad over $\mathcal{C}$. An $\mathcal{O}$-algebra in $\mathcal{C}$ is an object X in $\mathcal{C}$ and the data of a morphism of operads

$$\phi : \mathcal{O} \to \mathrm{End}_X.$$

Another example of operads are the commutative algebra operad.

Definition 12.16. Let $\mathcal{C}$ be any symmetric monoidal category with unit I. The symmetric operad Com has $\mathrm{Com}(n) = I$ for all n, the composition maps are unit isomorphisms, and the unit map is the identity map.

Operads and their algebras may be seen as a generalization of the commutative monoids due to the following proposition.

Proposition 12.17. A Com-algebra in $\mathcal{C}$ is a commutative monoid over $\mathcal{C}$.

Proof. See [Sar]. ■

We can also recover the notion of monoids from operads as follows.

Definition 12.18. Suppose $(\mathcal{C}, \otimes, I)$ is a symmetric monoidal category that has finite coproducts, and $\otimes$ preserves finite coproducts in both variables. We can define the **associative algebra operad** as Ass with

$$\mathrm{Ass}(n) := \bigsqcup_{\Sigma_n} I.$$

The unit map is given by the identity, and the composition is induced by the map

$$\Sigma_m \times \Sigma_{j_1} \times ... \times \Sigma_{j_m} \to \Sigma_j$$

$$(\sigma, \tau_1, ..., \tau_m) \mapsto \sigma_{j_1,...,j_m} \circ (\tau_1 \times ... \times \tau_m) \in \Sigma_j.$$

This is symmetric with the natural Σ_n action on the coproducts. We denote the underlying non-symmetric operad as $\overline{\mathrm{Ass}}$.

Proposition 12.19. An Assoc-algebra in $\mathcal{C}$ is a usual monoid in $\mathcal{C}$.

Proof. See [Sar]. ■

12.4 Operads and Spaces

Let us now restrict to the special case where we have $(\mathrm{Top}_*, \times, *)$ as the closed symmetric monoidal category. For an operad $\mathcal{O}$ over spaces, we call an $\mathcal{O}$-algebra also as an $\mathcal{O}$-space. We first observe that the associative algebra operad becomes extremely specified, as the coproduct is the wedge sum.

Proposition 12.20. The (non-symmetric) associative algebra operad Ass over spaces has $\mathrm{Ass}(n) = *$ for all n, and all maps are determined accordingly. An algebra over Ass here specifies an n-ary operations $X^{\times n} \to X$ exactly that satisfies associativity and unitality.

One might wonder what exactly would be the difference between the non-symmetric Ass over spaces and the commutative algebra operad $\mathrm{Com}(n)$, which also has

$$\mathrm{Com}(n) = *, n \geq 0.$$

The difference is that in Com we specify this to be a **symmetric operad**, which also requires one to specify the composition maps are equivariant with respect to permutation groups.

This distinction gives us the following.

Theorem 12.21. A space X is an algebra over Ass if and only if X admits a strictly unital and associative operation (i.e, a topological monoid). A space X is an algebra over Comm if and only if X admits a strictly unitial, associative, and commutative operation (i.e., a topological abelian monoid).

As described earlier, the loop spaces do not specify strict unitality/associativity/commutativity. But one could hope for the next best thing.

Definition 12.22. An $\mathcal{A}_\infty$-operad is a non-symmetric operad $\mathcal{O}$ over spaces such that $\mathcal{O}(n)$ is contractible for all n. A $\mathbb{E}_\infty$-operad is a symmetric operad $\mathcal{O}$ over spaces such that the action of each Σ_n on $\mathcal{O}(n)$ is free.

We can an algebra over $\mathcal{A}_\infty$-operad as an $\mathcal{A}_\infty$-space, and an algebra over an $\mathbb{E}_\infty$-algebra as an $\mathbb{E}_\infty$-space.

Recall a space X is group-like if $\pi_0(X)$ is a group. This definition does not really make sense without the data of some map $X^{\times 2} \to X$. However, if X is say an algebra over an operad $\mathcal{O}$, then the definition would make sense by taking π_0.

From here, we have that:

Theorem 12.23 (May's Recognition Principle, Version 1 [May72]. See also [Bel17]). A group-like $\mathcal{A}_\infty$ space X is equivalent to **the loop space of some space** Y. A group-like $\mathbb{E}_\infty$ space X is the same as an **infinite loop space**.

Another appeal of working with spaces is that we can define suitable notions of equivalences.

Definition 12.24. Two (non-symmetric or symmetric) operads $\mathcal{O}, \mathcal{P}$ are (weakly) equivalent in Top if there is a morphism of operads $\phi : \mathcal{O} \to \mathcal{P}$ such that each $\mathcal{O}(n) \to \mathcal{P}(n)$ is a (Σ_n-equivariant, if symmetric) (weak) homotopy equivalence. This defines a **homotopical category structure** on the category of operads over spaces.

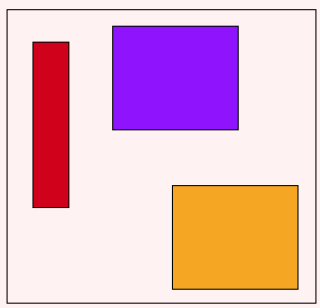
From this perspective, an $\mathcal{A}_\infty$-operad is any operad homotopy equivalent to Ass, and an $\mathbb{E}_\infty$-operad is any operad homotopy equivalent to Com.

In algebra, we just have the notion of associativity and commutativity. In spaces, however, it turns out we can interpolate between various notions of commutativity. They are described by what are called little-disk operads (first given by Boardman and Vogt [BV68]).

Definition 12.25. A little n-cube is a parallel axis affine embedding

$$c : [0,1]^n \to [0,1]^n, (t_1, ..., t_n) \mapsto (x_1 + a_n t_1, ..., x_n + a_n t_n)$$

for some fixed $(x_1, ..., x_n)$ and $(a_1, ..., a_n)$ with $a_i > 0, x_i \leq 0, x_i + a_i \leq 1$. Pictorially, here is an example of three 2-cubes:

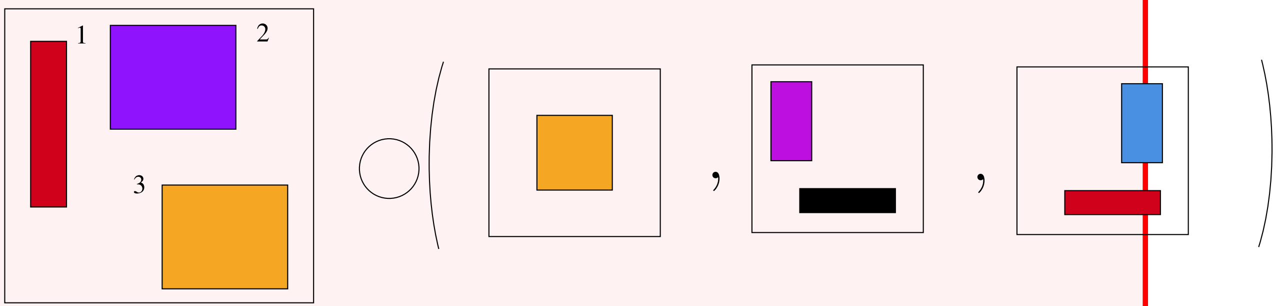


A collection of little n-cubes $c_1, ..., c_k$ are said to be **almost disjoint** if the interior of their images do not intersect.

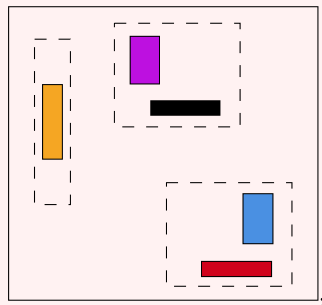
Definition 12.26. The **little n-cubes operad** is an operad of spaces $\mathcal{C}_n$. For each $k > 0$, $\mathcal{C}_n(k)$ is the space of k almost disjoint little n-cubes, topologized by the compact open topology. For $k = 0$, $\mathcal{C}_n(0))$ is the unique way to embed zero little-n-cubes (i.e., this is a point).

1. The unit map specifies the identity embedding $[0,1]^n \to [0,1]^n$.

2. Composition of little n-cubes are pictorially given by:



should give



and so on.

3. There is a Σ_n-action by reordering the embeddings.

We also define $\mathcal{C}_\infty$ with $\mathcal{C}_\infty(k) = \bigcup_n \mathcal{C}_n(k)$.

Proposition 12.27 ([MI22]). The spaces $\mathcal{C}_n(k)$ are equivalent to configuration space of ordered k-tuples of points in $\mathbb{R}^n$. It follows that $\mathcal{C}_\infty(k)$ are universal Σ_k-spaces, so in particular, $\mathcal{C}_\infty$ is a model for $\mathbb{E}_\infty$-operads!

Proposition 12.28. Let X be a space, then $\Omega^n X$ is an algebra over $\mathcal{C}_n$.

Proof Idea. Let $I = [0,1]$. We can view $\Omega^n X$ as the space of maps of pairs $(I^n, \partial I^n) \to (X, *)$. For each collection of k almost disjoint little cubes $c_1, ..., c_k$, and maps $f_i : (I^n, \partial I^n) \to (X, *)$, we can define a new map $f : (I^n, \partial I^n) \to (X, *)$ as follows:

1. For each little n-cube embedded in $[0,1]^n$, we let f be the map $f_i \circ c_i^{-1}$ on them.
2. This agrees on overlaps as they evaluate to $*$ on the boundary.
3. Then send every other point to $*$ as well.

More formally, this is the map

$$f(t_1, ..., t_n) = \begin{cases} f_i(c_i^{-1}(t_1, ..., t_n)), \text{ if } t \in \operatorname{im}(c_i) \text{ for some } i \\ *, \text{otherwise} \end{cases}$$

∎

Definition 12.29. An operad is $\mathbb{E}_n$ if it is weakly equivalent to the little n-cubes operad $\mathcal{C}_n$.

Over $\mathbb{C}_1$ this recovers our previous notion of associaitvity as well.

Proposition 12.30 (Proposition 3.8 of [MI22])**.** An operad of spaces is $\mathbb{E}_1$ if and only if it is equivalent to an $\mathcal{A}_\infty$-operad.

In particular, we have the following schematic implication

$$\mathbb{E}_\infty \implies ...\mathbb{E}_n \implies \mathbb{E}_{n-1} \implies ...\mathbb{E}_2 \implies \mathbb{E}_1 = \mathcal{A}_\infty.$$

Example 12.31. S^3 is an $\mathbb{E}_1$-space, as it is $\Omega\mathbb{H}P^\infty$, but is not $\mathbb{E}_2$.

The recognition principle from earlier has the following strengthening.

Theorem 12.32 (May's Recognition Principle, Version 2)**.** For $1 \leq n \leq \infty$, a group-like $\mathbb{E}_n$-space is the same as an n-fold loop space.

12.5 Operads and Spectra

Now we discuss notions of commutativity for spectra. It turns out that for an operad $\mathcal{O}$ over spaces, it is possible to define what it means when a spectrum X is an $\mathcal{O}$-algebra. There are different ways for doing this, which we briefly outline below.

Definition 12.33. Let $\mathcal{O}$ be an operad over spaces. Let X be a symmetric spectrum, an $\mathcal{O}$-algebra structure on X is a collection of equivariant maps

$$\mathcal{O}(n)_+ \wedge R^{\wedge n} \to R$$

for each $n \geq 0$ satisfying certain compatibility conditions with associativity and unitality. This is for example the definition in [BM11].

Remark 12.34. There is a more general definition one could use as well. For any closed symmetric monoidal category $\mathcal{C}$ enriched and tensored over spaces, one could make sense of what the following term means

$$X \otimes c \in \mathcal{C},$$

for a space X and an object $c \in \mathrm{obj}(\mathcal{C})$. For an operad $\mathcal{O}$ over spaces, this gives a natural way to generalize an $\mathcal{O}$-algebra in $\mathcal{C}$, as the endomorphism operad for X would be an operad of spaces. Here, the category of symmetric spectra is certainly enriched and tensored over spaces.

For (symmetric) spectra specifically, one could also take the suspension spectrum Σ^∞_+ of $\mathcal{O}$ to turn it into an operad of (symmetric) spectra. This actually does line up with the description above, as (symmetric) spectra are tensored over spaces precisely by smashing with the suspension spectrum of the space.

For an $\mathbb{E}_\infty$-ring spectrum A, one can also make sense of what it means to be an $\mathbb{E}_n$-A-algebra by replacing the base of smash product accordingly.

They have quite nice implications for the modules associated to them:

Theorem 12.35 (From [MI22], Theorem 3.7-3.8 of [Law20a]). Let R be a spectrum in a symmetric monoidal category $\widehat{\mathrm{Sp}}$,

1. An $\mathbb{E}_1$-algebra structure on R gives a well-defined category of left and right modules $\mathrm{LMod}_R, \mathrm{RMod}_R$.
2. An $\mathbb{E}_2$-algebra structure on R makes $\mathrm{ho}(\mathrm{LMod}_R) = \mathrm{ho}(\mathrm{RMod}_R)$, and a monoidal structure $\otimes_R$ on the homotopy category of left/right modules.
3. An $\mathbb{E}_3$-algebra structure on R extends the monoidal structure to be braided monoidal.
4. An $\mathbb{E}_4$-algebra structure on R extends this into a symmetric monoidal category.
5. An $\mathbb{E}_1$-algebra structure on R gives a well-defined monoidal category of bimodules.
6. If R is $\mathbb{E}_2$, then $\pi_*(R)$ is graded commutative. In particular, if R has homotopy groups concentrated in even degrees, then $\pi_*(R)$ is a commutative ring.

Finally, we state the rectification theorem.

Theorem 12.36 (Rectification Theorem). Let $\widehat{\mathrm{Sp}}$ be a symmetric monoidal category of spectra. If X is an $\mathbb{E}_1$-algebra (which is equivalent to an $\mathcal{A}^\infty$-algebra) in $\widehat{\mathrm{Sp}}$, then there is a monoid $Y \in \widehat{\mathrm{Sp}}$ such that $X \simeq Y$. Similarly, if X is an $\mathbb{E}_\infty$-algebra in $\widehat{\mathrm{Sp}}$, then there is a commutative monoid $Y \in \widehat{\mathrm{Sp}}$ such that $X \simeq Y$.

13 Week 13: Localizations by Abbigael Arseneau and Jackson Wages

Today we will be talking about localizations in Spectra. Jackson Wages will talk about the first half, and Abbigael Arseneau will talk about the second half. The syllabus recommends [Kou13] as the source for this lecture. The notetaker additionally recommends [Law20b] as a helpful reference.

The outline of Jackson's part is as follows:

1. We will first define what E_*-local, E_*-equivalences mean.
2. We will then give some examples of E-localization.
3. Then we will formalize the localization functor explicitly.
4. Finally we will talk about some computations of localizations in practice.

The outline of Abbigael's part is as follows:

1. Briefly outline some ideas behind the construction of Bousfield localzation.
2. Explain the fracture square theorem for localizations.
3. Prove the arithmetic fracture square as an example.

13.1 Introduction to Localizations of Spectra by Jackson Wages

Motvation: We can study spectra with homology/cohomology. Recall that every spectra E defines:

- A generalized cohomology theory as
$$E^n X \coloneqq \pi_{-n} F(X, E).$$
- A generalized homology theory as
$$E_n X \coloneqq \pi_n (X \wedge E).$$

We will mostly be talking about **generalized homology theories** today. Similar to the case in algebra, we can study spectra up to E_*-equivalence.

Definition 13.1. A map of spectra $f : X \to Y$ is a E_***-equivalence** when the induced map $E_* f : E_*(X) \to E_*(Y)$ is an isomorphism.

We also have the notion of localization, as follows.

Definition 13.2. A spectrum A is E_*-acyclic if $E_* A = 0$. A spectrum X is E_*-local if for all E_*-acyclic A, the homotopy classes of maps $[A, X]_* = 0$.

Proposition 13.3. A map $f : X \to Y$ is an E_*-equivalence if and only if the fiber of f is E_*-acyclic.

Proof. We have a long exact sequence in E_*-homology given by

$$\ldots \to E_n \operatorname{fib}(f) \to E_n X \xrightarrow{f_*} E_n Y \to E_{n-1} \operatorname{fib}(f) \to \ldots$$

Clearly f being an E_*-equivalence is equivalent to $E_* \operatorname{fib}(f) = 0$. ■

Proposition 13.4. A spectrum Z is E_*-local if and only if every E_*-equivalence $f : X \to Y$ induces an isomorphism $[X, Z] \cong [Y, Z]$.

Proof. In general for a fiber sequence $F \to X \xrightarrow{f} Y$, there is a long exact sequence of the form

$$\ldots \to [\Sigma F, Z] \to [Y, Z] \to [X, Z] \to [F, Z] \to \ldots$$

Suppose Z is E_*-local. By Proposition 13.3, F is E_*-acyclic, so $[F, Z] = 0$, so $\Sigma F = \Sigma\mathbb{S} \wedge F$ and is hence also E_*-acyclic, so we have

$$0 \to [Y, Z] \to [X, Z] \to 0$$

is an isomorphism.

Conversely, let F be any E_*-acyclic spectrum. Then the 0-map $F \to F$ is a E_*-equivalence, and hence induces an isomorphism $[F, Z] \to [F, Z]$. This means the zero map is an isomorphism on $[F, Z]$, so it follows that $[F, Z] = 0$, and hence Z is E_*-local. ∎

We are now equipped to define the notion of localization as follows.

Definition 13.5. An E_*-localization of a spectrum X is an E_*-equivalence $X \to L_E X$ where $L_E X$ is E_*-local. Note that the map $X \to L_E X$ is initial among all maps $X \to Y$ where Y is E-local.

Definition 13.6. An E_*-localization functor is a functorial choice of E_*-localizations.

It is not obvious from the definition that such $L_E X$ exists, but we will discuss later on how this is the case.

Theorem 13.7 (Bousfield)**.** For any spectrum E, there exists a localization functor

$$X \mapsto [X \to L_E X].$$

Theorem 13.8 (Whitehead Theorem for E_*-local Spectra)**.** If $f : X \to Y$ is an E_*-equivalence of E_*-local spectra, then f is a stable equivalence.

Proof. By Proposition 13.4, we know that $[X, Z] \cong [Y, Z]$ for all E-local spectrum Z. By Yoneda's lemma, this means that X and Y are equivalent in the full subcategory of E-local spectra, and hence in spectra. ∎

We also state the following structural proposition on E_*-local spectraa, which appears as Proposition 3.6 and 3.7 of [Kou13].

Proposition 13.9. E_*-local spectra are closed under shifts, products, retracts, and cofibers. In fact, E_*-local spectra are closed under all limits (as, $[X, -]$ preserves limits).

Proof Comment. The proof for the cofiber is an interesting exercise in Five Lemma. ∎

Proposition 13.10. If L_E is any localization functor, then L_E preserves shifts, wedges, and homotopy cofibers.

13.2 Examples of Localization by Jackson Wages

Here we look at some examples of localizations.

Example 13.11. Let $E = H\mathbb{Q}$ and M be a $H\mathbb{Q}$-local spectrum. It will be the case that $\pi_*(M)$ is a $\mathbb{Q}$-vector space. Indeed, the multiplication by n map $\times n : \mathbb{S} \to \mathbb{S}$ gives a cofiber sequence

$$\mathbb{S} \xrightarrow{\times n} \mathbb{S} \to \mathbb{S}/n.$$

Taking the underlying cohomology on M, we get

$$\ldots \leftarrow [\mathbb{S}/n, M]_{*-1} \leftarrow [\mathbb{S}, M]_* \xleftarrow{\times n} [\mathbb{S}, M]_* \leftarrow [\mathbb{S}/n, M]_* \leftarrow \ldots$$

Observe that $\mathbb{S}/n$ is $H\mathbb{Q}$-acyclic since the map $\times n : \mathbb{S} \to \mathbb{S}$ is an $H\mathbb{Q}$-equivalence. Since M is $H\mathbb{Q}$-local, we have that

$$\times n : [\mathbb{S}, M]_* \to [\mathbb{S}, M]_* \text{ is an isomorphism.}$$

It follows that $\pi_*(M) \xrightarrow{\times n} \pi_*(M)$ is invertible for any n, so $\pi_*(M)$ is a rational vector space.

Let us look at an example where we put more structure on E.

Proposition 13.12. Let R be a ring spectrum and M be an R-module, then M is R-local.

Proof. Let A be an R-acyclic spectrum, we wish to show that $[A, M] = 0$. Write $i : \mathbb{S} \to R$ to be the unit map. For any map $f : A \to M$, we have a commutative diagram of the form:

$$\begin{tikzcd}
\mathbb{S} \wedge A \simeq A \arrow[r, "f"] \arrow[d, "i \wedge \mathrm{id}_A"'] & M \simeq \mathbb{S} \wedge M \arrow[d, "i \wedge \mathrm{id}_M"] \arrow[dr, "\mathrm{id}_M"] & \\
R \wedge A \arrow[r, "\mathrm{id}_R \wedge f"'] & R \wedge M \arrow[r, "\mu\text{,action map}"'] & M
\end{tikzcd}$$

Since A is by definition R-acyclic, we have that $R \wedge A \simeq *$, so this diagram becomes:

$$\begin{tikzcd}
\mathbb{S} \wedge A \simeq A \arrow[r, "f"] \arrow[d, "0"'] & M \simeq \mathbb{S} \wedge M \arrow[d, "i \wedge \mathrm{id}_M"] \arrow[dr, "\mathrm{id}_M"] & \\
* \arrow[r, "0"'] & R \wedge M \arrow[r, "\mu\text{,action map}"'] & M
\end{tikzcd}$$

This means that $f = \mathrm{id}_M \circ f$ is equal to $\mu \circ 0 \circ 0 = 0$. Thus, we conclude that $[A, M] = 0$, and M is hence R-local. ∎

Definition 13.13. A localization is **smashing** if $L_E X = L_E \mathbb{S} \wedge X$.

Example 13.14. The localization at $H\mathbb{Q}$ is in fact smashing. In this case, $L_{H\mathbb{Q}}\mathbb{S} \simeq H\mathbb{Q}$, and hence

$$L_{H\mathbb{Q}} X = H\mathbb{Q} \wedge X.$$

Example 13.15. Let p be a prime number. Localization at $\mathbb{S}/p$ is constructed as a homotopy colimit

$$\mathrm{hocolim}_n(X/p^n), \text{ which looks like the construction of } p\text{-adic integers } \mathbb{Z}_p.$$

Importantly, this is not a smashing localization. However, by Proposition 2.5 of [Bou79], if $\pi_n(X)$ are finitely generated, then $\pi_n(L_{\mathbb{S}/p}X) \cong \pi_n(X)_p^\wedge := \pi_n(X) \otimes \mathbb{Z}_p$. We often write $X_p^\wedge$ to denote $L_{\mathbb{S}/p}X$, and this is called the **p-completion of X**.

We conclude this part with an important theorem about Bousfield localization of connective spectra.

Definition 13.16. Let G be an abelian group. The **Moore spectrum** SG of G is a spectrum characterized by the properties that:

1. $\pi_k(SG) = 0$ for $k < 0$.
2. $\pi_0(SG) = G$.
3. $H_{>0}(SG, \mathbb{Z}) = 0$.

Example 13.17. $S\mathbb{Q}$ is $H\mathbb{Q}$, and $\mathbb{S}/p$ is $S\mathbb{Z}/p$.

Theorem 13.18 (Bousfield, See Theorem 4.6 of [Kou13])**.** Let X be a connective spectra and E also be a connective spectra. Then

$$L_E X \simeq L_{S\pi_0 E} X.$$

Here $S\pi_0 E$ refers to the **Moore spectrum** of the group $\pi_0 E$.

Since $\mathbb{S}/p$ is $S\mathbb{Z}/p$ and $\pi_0(\mathbb{S}/p) = \mathbb{F}_p$, we have that:

Corollary 13.19. Let X be a connective spectra, then $X_p^\wedge \simeq L_{H\mathbb{F}_p} X$.

13.3 Constructing the Localization Functor by Abbigael Arseneau

Motivation: Recall for $A \in \mathrm{Ab}$, we can localized at a prime (p) that fits in an exact sequence

$$(p)A \to A \to A_{(p)},$$

where $A_{(p)}$ is p-local.

Notation: We use X_E to denote $L_E X$ interchangeably.

Theorem 13.20 (Bousfield, Theorem 1.1 of [Bou79])**.** Given $E \to \mathrm{Ho}(\mathrm{Sp})$, there is a (co)fiber sequence

$${}_E X \xrightarrow{\theta_X} X \xrightarrow{\eta_X} X_E,$$

where X_E is E-local and ${}_E X$ is E-acyclic. We call η_X the E-localization of X.

Remark 13.21. For a spectrum E, we can look at Sp_E, the full subcategory of E-local spectra. Similarly, we have a full subcategory ${}_E\,\mathrm{Sp} \subseteq \mathrm{Sp}$ of E-acyclic spectra.

1. The inclusions $\mathrm{Sp}_E \hookrightarrow \mathrm{Sp}$ has a **left adjoint**, which is η (or L_E) - the E-localization.
2. The inclusion ${}_E\,\mathrm{Sp} \hookrightarrow \mathrm{Sp}$ has a **right adjoint**, which is θ as above.

Let us make some observations/properties from Theorem 13.20.

1. η_X is a localization functor. Indeed, this clearly follows because the fiber ${}_EX$ is E-acyclic.
2. The functor $L_E : \mathrm{Sp} \to \mathrm{Sp}_E$ is an **idempotent functor**, i.e., $L_E(L_E(X)) \simeq L_EX$. (This follows from the E_*-Whitehead theorem by looking at $\eta_{L_E(X)}$).
3. For a spectrum X, the localization functor η_X is **initial** among all maps from $X \to E$-local spectrum.
4. For a spectrum X, the map θ_X is terminimal among all maps from E-acyclic spectrum $\to X$.

Let us prove the initiality of the localization functor as an example.

Proof. Suppose $f : X \to Y$ is a map where Y is E-local. Since η_X is an E-equivalence, we get an isomorphism $[X, Y] \cong [L_EX, Y]$, which gives the desired deduction. ■

13.4 Bousfield Classes and Arithmetic Fracture Square by Abbigael Arseneau

Definition 13.22. We say E, E' are Bousfield equivalent if for all $X \in \mathrm{Sp}$, X is E-acyclic if and only if X is E'-acyclic.

We write $\langle E\rangle$ to denote all spectra that are E-acyclic. We give a partial ordering on all Bousfield equivalent classes where $\langle E\rangle \leq \langle F\rangle$ if X being F-acyclic implies X is E-acyclic.

Note: This in fact forms a lattice with the minimum being $\langle *\rangle$ and the maximum being $\langle \mathbb{S}\rangle$.

Here we record a very useful fact about these localizations.

Lemma 13.23. (1) Suppose $\langle D\rangle \leq \langle E\rangle$ (i.e., Y being E-acyclic implies Y is D-acyclic), then for any spectrum X, there is a natural map $L_EX \to L_DX$.

(2) Furthermore, this map is an L_D-equivalence, i.e, the map

$$L_DL_EX \to L_DL_DX \simeq L_DX \text{ is an equivalence.}$$

This is a consequence of the fact that the map

$$X \to L_EX \text{ is an } L_D\text{-equivalence,}$$

and noting that $X \to L_DX$ is clearly an L_D-equivalence by idempotence.

Proof. For (1): recall the map $X \to L_EX$ is among maps $X \to Y$ where Y is E-local. To induce the map $L_EX \to L_DX$, it suffices to show that L_DX is E-local. Suppose Y is an E-acyclic spectrum, then since $\langle D\rangle \leq \langle E\rangle$, Y is D-acyclic, so

$$[Y, L_DX] = 0,$$

and hence it follows that L_DX is E-acyclic.

For (2): It suffices to show that the map $X \to L_EX$ is an L_D-equivalence. Indeed, the fiber of $X \to L_EX$ is E-acyclic, and hence D-acyclic. The D-localization of a D-acyclic spectra is trivial, so the map $L_DX \to L_DL_EX$ is an equivalence. ■

Now we explain an important result in localization theory known as the fracture square theorem.

Theorem 13.24 (The Fracture Square Theorem)**.** Let E, F be spectra such that for all spectra Y,

$$E \wedge Y \simeq 0 \implies E \wedge L_F Y \simeq 0$$

then for all spectra X. Note this is satisfied the L_F is a **smashing localization**. Then there is a homotopy pullback square of the form:

$$\begin{tikzcd} L_{E\vee F}X \arrow[r, "\eta_E"] \arrow[d, "\eta_F"'] & L_E X \arrow[d, "\eta_F"] \\ L_F X \arrow[r, "L_F(\eta_E)"'] & L_F L_E X \end{tikzcd}$$

Here the maps $\eta_E : L_{E\vee F}X \to L_E X$ and $\eta_F : L_{E\vee F}X \to L_F X$ are obtained by noting the map $X \to L_{E\vee F}X$ is both an E-equivalence and F-equivalence, and then apply the terminality condition.

Remark 13.25. In particular, this theorem also holds when for all spectra Y, $E \wedge L_F Y \simeq 0$.

Here the notetaker (Mattie Ji) expands the lecture notes with a proof following [Str12] and [Gui24]. The tex below is taken from some talk notes the notetaker wrote before.

Proof. Take P to be the pullback for the triangle above and consider the map:

$$\begin{tikzcd} X \arrow[dr, dashed] \arrow[drr, "\eta_E"] \arrow[ddr, "\eta_F"'] & & \\ & P \arrow[r] \arrow[d] & L_E X \arrow[d, "\eta_F"] \\ & L_F X \arrow[r, "L_F(\eta_E)"'] & L_F L_E X \end{tikzcd}$$

We claim that the map $X \to P$ is indeed a $(E \vee F)$-localization.

We first note that P has to be $E \vee F$-local, this requires shows that any map $Y \to P$ where Y is $E \vee F$-acyclic is null-homotopic. Indeed, recall that every Mayer Vietoris sequence corresponds exactly to a homotopy pullback. For completeness we illustrate the idea here with the following lemma.

Lemma 13.26. Consider a homotopy pullback diagram

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "f'"] \\ C \arrow[r, "g'"'] & D \end{tikzcd}$$

Then there is a Mayer-Vietoris sequence of the form

$$... \to [\Sigma Y, D] \to [Y, A] \to [Y, B] \oplus [Y, C] \to [Y, D] \to ...$$

Now we take this Mayer-Vietoris and apply this to Y, which is $E \vee F$-acyclic, so we have a long exact sequence

$$... \to [Y, P] \to [Y, L_E X] \oplus [Y, L_F X] \to [Y, L_F L_E X] \to ...$$

Since Y is $E \vee F$-acyclic, clearly $[Y, L_E X]$ and $[Y, L_F X] \to [Y, L_F L_E X]$ are zero, suspensions do not affect acyclicity, so we conclude that $[Y, P] = 0$. Thus, P is $E \vee F$-acyclic.

Now we wish to check that the map $X \to P$ is an $E \vee F$-equivalence. It suffices to check this is both an F-equivalence and an E-equivalence.

For F - Observe the map $P \to L_F X$ is an F-equivalence because the pullback square of an F-equivalence is an equivalence, and the map $\eta_F : X \to L_F X$ is an F-equivalence. By the 2-out-of-3 property of isomorphisms, it means $X \to P$ is an F-equivalence.

For E, the fiber B of $X \to L_E X$ is clearly E-acyclic. The condition we have tells us that $L_F B$ is E-acyclic, and we have a fiber sequence (since L_F is exact)

$$L_F B \to L_F X \xrightarrow{L_F(\eta_E)} L_F L_E X$$

so we conclude the map $L_F(\eta_E)$ is an E-equivalence. Then the same argument with 2-out-of-3 shows that $X \to P$ is an E-equivalence. ■

As an application when $F = H\mathbb{Q}, E = \bigvee_p \mathbb{S}/p$, and since $L_{H\mathbb{Q}}$-localization is smashing, we have the following corollary.

Theorem 13.27 (Arithmetic Fracture Square)**.** For any $X \in \mathrm{Ho}(\mathrm{Sp})$, we have a pushout/pullback square:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & (\prod_{\text{prime p}} X_p^\wedge) \arrow[d] \\ X_\mathbb{Q} \arrow[r] & (\prod_{\text{prime p}} X_p^\wedge)_\mathbb{Q} \end{tikzcd}$$

Finally, let us look at the arithmetic fracture square on an example.

Example 13.28. Take $X = \mathbb{Z}/p^\infty$ which is a colimit $\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow ....$ As an abelian group, the following is not a pullback square in abelian groups:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & (\prod_{\text{prime p}} X_p^\wedge) \arrow[d] \\ X_\mathbb{Q} \arrow[r] & (\prod_{\text{prime p}} X_p^\wedge)_\mathbb{Q} \end{tikzcd}$$

In $\mathrm{Ho}(\mathrm{Sp})$ however, we note that $\mathbb{Z}/p^\infty = \mathrm{cofib}(\mathbb{Z} \xrightarrow{\frac{1}{p}} \mathbb{Z}[\frac{1}{p}])$. In this case, we have tthat

$$L_\mathbb{Q}(X) = 0 \text{ as } L_\mathbb{Q}(\mathbb{Z}/p^n) = 0.$$

Now for $q \neq p$, we can deduce that $X_q^\wedge = *$. For p, we have a long exact sequence

$$...L_{\mathbb{S}/p}(\mathbb{Z}) \to L_{\mathbb{S}/p}(\mathbb{Z}[\frac{1}{p}]) = 0 \to L_{\mathbb{S}/p}(\mathbb{Z}/p^\infty) \to L_{\mathbb{S}/p}(\Sigma\mathbb{Z}) \to L_{\mathbb{S}/p}(\Sigma\mathbb{Z}[\frac{1}{p}]) = 0 \to ...$$

Here $L_{\mathbb{S}/p}(\Sigma\mathbb{Z}[\frac{1}{p}]) = 0$ because localization commutes with suspension. From here we can deduce that

$$L_{\mathbb{S}/p}(\mathbb{Z}) = \mathbb{Z}_p \text{ and hence } X_p^\wedge \simeq \Sigma\mathbb{Z}_p.$$

Finally $(\Sigma\mathbb{Z}_p)_\mathbb{Q} = \Sigma\mathbb{Q}_p$. The following is indeed a pullback (and cofiber sequence) in spectra:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ \Sigma\mathbb{Z}_p \arrow[r] & \Sigma\mathbb{Q}_p \end{tikzcd} .$$

Morally the distinction between why the pullback square worked in spectra but not abelian groups is that abelian groups do not have a notion of suspension.

13.5 Problem Sessions Addendum by Kiran Luecke

In the category Ab, there is also a fracture square as follows. Let $A \in \mathrm{Ab}$, there is a commutative square of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \prod_p A_p^\wedge \arrow[d] \\ A \otimes \mathbb{Q} \arrow[r] & (\prod_p A_p^\wedge) \otimes \mathbb{Q} \end{tikzcd}.$$

This is however not always a pullback. A counter-example would be $\mathbb{Z}/q^\infty$ discussed in lecture, for some fixed prime q.

Proposition 13.29. The commutative square

$$\begin{tikzcd} \mathbb{Z}/q^\infty \arrow[r] \arrow[d] & \prod_p (\mathbb{Z}/q^\infty)_p^\wedge \arrow[d] \\ (\mathbb{Z}/q^\infty) \otimes \mathbb{Q} \arrow[r] & (\prod_p (\mathbb{Z}/q^\infty)_p^\wedge) \otimes \mathbb{Q} \end{tikzcd}$$

is not a pullback in Ab.

Proof. Indeed there is a short exact sequence of groups

$$0 \to \mathbb{Z} \xrightarrow{1 \mapsto 1/q} \mathbb{Z}[\frac{1}{q}] \to \mathbb{Z}/q^\infty \to 0$$

Since rationalization is tensoring with a flat module, this preserves the exact sequence and turns $\mathbb{Z} \to \mathbb{Z}[\frac{1}{p}]$ to $\mathbb{Q} \xrightarrow{\sim} \mathbb{Q}$, so the rationalization of $A = \mathbb{Z}/q^\infty$ is 0.

On the other hand $\prod_p (\mathbb{Z}/q^\infty)_p^\wedge = 0$ as well. This can be seen by splitting to the case $p = q$ and $p \neq q$ and applying the limit definition to deduce the result. Thus we have a pullback square in Ab whose top-right and bottom-left sides are 0 and the two other groups are non-zero. This gives a contradiction. ∎

Remark 13.30. Morally, the arithmetic fracture square gives the ability to recover an object from its rationalization and completions, glued along their intersections. This does not always work in Ab, but this does work in Sp. The key is that in spectra,

$$\prod_p (\mathbb{Z}/q^\infty)_p^\wedge \text{ is not longer zero but equal to } \Sigma\mathbb{Z}_p.$$

14 Week 14: Chromatic Homotopy Theory by Courtney Hauf and Vyom

In the last week of the eCHT class on stable homotopy theory, we will be doing a brief treatise of **chromatic homotopy theory**. Vyom will talk about the first half, and Courtney Hauf will talk about the second half.

14.1 MU and Related Constructions by Vyom

The purpose of this part is to introduce the complex-cobordism spectrum MU, Brown-Petersen spectrum BP, the truncated BP $\mathrm{BP}\langle n\rangle$, the Johnson-Wilson E-theory $E(n)$, and the Morava K-theory $K(n)$.

Definition 14.1. Recall that for B a paracompact space and a vector bundle $\pi : E \to B$, we can equip E with a metric. From here we define:

1. The **disk bundle** as
$$D(E) = \{e \in E \mid ||e|| \leq 1\}.$$
2. The **sphere bundle** as
$$S(E) = \{e \in E \mid ||e|| = 1\}.$$
3. Note that $S(E) \subseteq D(E)$, we define the **Thom space** as
$$T(E) \coloneqq D(E)/S(E).$$

Recall the following fact from vector bundles.

Proposition 14.2. Let ϵ^n be the trivial bundle of rank n over B, then
$$T(\epsilon^n \oplus E) \cong \Sigma^n T(E).$$

Recall, the classifying space $\mathrm{BU}(n)$ of the unitary group $U(n)$ has a universal bundle $\gamma_n \to \mathrm{BU}(n)$. From here we define the spectrum MU as follows.

Definition 14.3. We define a spectrum MU whose spaces are
$$\mathrm{MU}_{2n} = T(\gamma_n) \text{ and } \mathrm{MU}_{2n+1} = \Sigma T(\gamma_n),$$
and the structure map $\Sigma\,\mathrm{MU}_{2n} \to \mathrm{MU}_{2n+1}$ is given by the identity, the map $\mathrm{MU}_{2n+1} = \Sigma T(\gamma_n) \to \mathrm{MU}_{2n+2} = T(\gamma_{n+1})$ is given as follows.

1. $\Sigma^2\,\mathrm{Th}(\gamma_n)$ is the Thom space of $\gamma_n \oplus \mathbb{C}$ by Proposition 14.2.
2. $\gamma_n \oplus \mathbb{C}$ is a bundle over $BU(n)$. Consider the inclusion map $i : BU(n) \to BU(n+1)$ inducing by sending a unitary matrix A to $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$.
3. In this case, γ_{n+1} on $BU(n+1)$ pulls back to the bundle $\gamma_n \oplus \mathbb{C}$ on $BU(n)$ via this map.
4. This gives a morphism of vector bundles $\gamma_n \oplus \mathbb{C} \to \gamma_{n+1}$, and now use the functoriality of the Thom space construction.

It is a very deep theorem of Milnor (and independently, Novikov), that the following holds.

Theorem 14.4 (Milnor, Novikov). $\pi_*(\mathrm{MU}) = \mathbb{Z}[x_1, x_2, ..., x_n, ...]$ where $|x_n| = 2n$.

Now we wish to consider the p-localization of MU. We first explain what p-localization means.

Definition 14.5. We define the p-local sphere spectrum $\mathbb{S}_{(p)}$ as the homotopy colimit of

$$\mathbb{S} \xrightarrow{\cdot p_1} \mathbb{S} \xrightarrow{\cdot p_1 p_2} \mathbb{S} \xrightarrow{\cdot p_1 p_2 p_3} ...$$

where $p_1, p_2, ..., p_n, ...$ is any sequence of all prime numbers other than p.

Equivalently, $\mathbb{S}_{(p)}$ is the Bousfield localization of $\mathbb{S}$ at the Moore spectrum $M\mathbb{Z}_{(p)}$, which is what p-localization means usually. It turns out p-localization is smashing, so we can take as a definition or a fact that for a spectrum X, its p-localization is

$$X_{(p)} := X \wedge \mathbb{S}_{(p)}.$$

Lemma 14.6. $\pi_*(\mathrm{MU}_{(p)}) = \mathbb{Z}_{(p)}[x_1, x_2, ...]$.

Proof. Since p-localization is given by the smashing product with $\mathbb{S}_{(p)}$ and smashing product commutes with colimits, evidently,

$$MU_{(p)} = MU \xrightarrow{\cdot p_1} MU \xrightarrow{\cdot p_1 p_2} MU \xrightarrow{\cdot p_1 p_2 p_3} ...$$

Since homotopy groups commute with sequential colimits (by Lemma 5.39), we have that

$$\pi_*(MU_{(p)}) = \pi_*(\mathrm{MU}) \otimes \mathbb{Z}_{(p)} = \mathbb{Z}_{(p)}[x_1, x_2, ...].$$

■

Now we consider a general construction.

Definition 14.7. Let R be a ring spectrum, a sequence $(f_1, f_2, ...)$ in $\pi_*(R)$ is called a **regular sequence** if for all $n \geq 1$, multiplication by f_n is injective on $\pi_* R/(f_1, ..., f_{n-1})$.

On the other hand, we can take an iterated cofiber of the self-maps $f_1, f_2, ..., f_n$ to form the spectrum $R/(f_1, ..., f_n)$. It is a fact that if the sequence is regular, then

$$\pi_*(R/(f_1, ..., f_n)) = \pi_* R/(f_1, ..., f_n).$$

Definition 14.8. The Brown-Petersen spectrum BP is defined as

$$\mathrm{MU}_{(p)} / \langle x_i, i \neq p^k - 1 \text{ for some } k \rangle.$$

This is a regular sequence and as a consequence, we have that:

Corollary 14.9. $\pi_* \mathrm{BP} = \mathbb{Z}_{(p)}[v_1, ..., v_i, ...]$ where $|v_i| = 2(p^i - 1)$. (Here we reindexed them to v_i's).

Definition 14.10. The truncated Brown Petersen spectrum $\mathrm{BP}\langle n\rangle$ is defined by quotienting the regular sequence $(v_{n+1}, v_{n+2}, ...)$.

Corollary 14.11. $\pi_* \mathrm{BP}\langle n\rangle = \mathbb{Z}_{(p)}[v_1, ..., v_n]$

To get to Johnson-Wilson E-theory, we need the following construction.

Definition 14.12. Let R be a ring spectrum and $f \in \pi_n(R)$, we define $R[f^{-1}]$ as the homotopy colimit of the form

$$R \xrightarrow{f} \Sigma^{-n} R \xrightarrow{f} \Sigma^{-2n} R \to ...$$

Since sequential colimits commute with homotopy groups, this has the effect of

$$\pi_*(R[f^{-1}]) = \pi_*(R)[f^{-1}].$$

Definition 14.13. We define the **Johnson-Wilson E-theory** $E(n)$ as $\mathrm{BP}\langle n\rangle[v_n^{-1}]$.

Corollary 14.14. $\pi_*(E(n)) = \mathbb{Z}_{(p)}[v_1, ..., v_{n-1}][v_n^{\pm 1}]$.

Finally, we define Morava K-theory $K(n)$ as follows.

Definition 14.15. We define $K(n)$ as the quotient of $E(n)$ by the regular sequence $(p, v_1, ..., v_n)$.

Corollary 14.16. We have that $\pi_*(K(n)) = \mathbb{F}_p[v_n^{\pm 1}]$ where $|v_n| = 2(p^n - 1)$.

By convention, we write $K(0)$ and $E(0)$ as both $H\mathbb{Q}$.

14.2 The Chromatic Tower by Courtney Hauf

We have covered what $K(n)$ and $E(n)$ are. We state the following deep theorem of Ravenel without proof.

Theorem 14.17 (Ravenel)**.** $E(n)$ and $E(n-1) \vee K(n)$ are Bousfield equivalent.

It follows that a spectrum that is $E(n-1)$-local is $E(n)$-local. From now on in this section, we let X be a spectrum and fix a prime p.

Definition 14.18. Consider the $E(n)$-localization of X, $L_{E(n)}X$. Since X is $E(n)$-local, it is $E(n+1)$-local, by the universal property of localization, this induces a map

$$L_{E(n+1)}X \to L_{E(n)}X.$$

These maps assemble to a tower

$$X \to ... \to L_{E(n+1)}X \to L_{E(n)}X \to ... \to L_{E(0)}X = X \otimes \mathbb{Q}.$$

This is called the **chromatic tower** of X.

The following is a very deep theorem of Hopkins and Ravenel.

Theorem 14.19 (Chromatic Convergence Theorem by Hopkins-Ravenel)**.** If X is finite p-local, then $X \simeq \operatorname{holim}_n L_{E(n)}X$. In particular,

$$\mathbb{S}_{(p)} \simeq \operatorname{holim}_n L_{E(n)}\mathbb{S}.$$

Remark 14.20. $H\mathbb{F}_p$ is not finite but p-local. This is a counter-example to the chromatic convergence theorem.

The **upshot** of the **Chromatic Convergence Theorem** is that to understand the p-local parts of X, it suffices to under the $E(n)$-local parts of X.

By an extension of the fracture square theorem that we discussed last week, we have what is called the **chromatic fracture square**.

Theorem 14.21. There is a homotopy pullback of the form:

$$\begin{tikzcd} L_{E(n)}X \arrow[r] \arrow[d] & L_{E(n-1)}X \arrow[d] \\ L_{K(n)}X \arrow[r] & L_{E(n-1)}L_{K(n)}X \end{tikzcd}$$

The reason why this fracture square holds is because localization by $E(k)$ is smashing for any k. This is a theorem called the smash product therem.

Theorem 14.22 (The Smash Product Theorem)**.** $L_{E(k)}$ is a smashing localization, and hence

$$X \simeq L_{E(k)}\mathbb{S} \otimes X.$$

In particular we can rewrite the **chromatic tower** to:

$$X \to \ldots \to L_{E(n)}\mathbb{S} \otimes X \to \ldots \to L_{E(1)}\mathbb{S} \otimes X \to L_{E(0)}\mathbb{S} \otimes X = X \otimes \mathbb{Q}.$$

We can also **rewrite the chromatic fracture square** into:

$$\begin{tikzcd} L_{E(n)}\mathbb{S} \otimes X \arrow[r] \arrow[d] & L_{E(n-1)}\mathbb{S} \otimes X \arrow[d] \\ L_{K(n)}X \arrow[r] & L_{E(n-1)}\mathbb{S} \otimes L_{K(n)}X \end{tikzcd}$$

Morally, we can think of the $K(n)$-localizations as a measurement of how far away the two terms are in the chromatic tower.

In general, computing the $K(n)$-local sphere $L_{K(n)}\mathbb{S}$ has been quite difficult. For $n = 1$, this is known. For $n = 2$, there has been a lot of progress in understanding this. For $n > 2$, the area is very wide open. Recently, there is a very impressive theorem that computes the rationalization of the $K(n)$-local sphere.

Theorem 14.23 ([BSSW25])**.** As a graded $\mathbb{Q}$-algebra, $\pi_*(L_{K(n)}\mathbb{S} \otimes \mathbb{Q}) = \bigwedge_{\mathbb{Q}_p}(\zeta_1, \ldots, \zeta_n)$ is the exterior $\mathbb{Q}_p$-algebra with generators $\zeta_1, \ldots, \zeta_n$ such that $|\zeta_i| = 1 - 2i$.

A Miscellaneous Lemmas

The correctness of the following proofs are not entirely verified.

Lemma A.1. Let $S^3 = \underbrace{(S^1)}_a \wedge \underbrace{(S^1)}_b \wedge \underbrace{(S^1)}_c$ and $f : S^3 \to S^3$ be the self-map that permutes the copies of S^1's in the order

$$a \mapsto b, b \mapsto c, c \mapsto a.$$

Then $\deg(f) = 1 \in \pi_3(S^3) \cong \mathbb{Z}$ and hence f is homotopic to the identity.

Proof. We first observe that the swap map $\mathrm{swap} : S^1 \wedge S^1 \to S^1 \wedge S^1$ induces the multiplication by -1 map on $\pi_2(S^2) = \mathbb{Z}$. One way to see this is to take an explicit point-set model of $S^1 \wedge S^1$ as $S^1 \times S^1/(S^1 \vee S^1)$ (which gets more complicated for n-fold smash products). The quotient map

$$q : S^1 \times S^1 \to S^1 \wedge S^1$$

is equivariant with respect to the C_2-action (swapping the two components) on both sides and in particular implies that

$$\begin{tikzcd} H_2(S^1 \times S^1) \arrow[r, "q_*"] \arrow[d, "\mathrm{swap}"'] & H_2(S^1 \wedge S^1) \arrow[d, "\mathrm{swap}"] \\ H_2(S^1 \times S^1) \arrow[r, "q_*"'] & H_2(S^1 \wedge S^1) \end{tikzcd}$$

The map q_* is an isomorphism, and hence we see it suffices to show $\mathrm{swap} : H_2(S^1 \times S^1) \to H_2(S^1 \times S^1)$ is multiplication by -1. This can be verified explicitly, or by noting this swap is an orientation reversing homeomorphism.

Now we observe that the permutation in the lemma can be decomposed into the composition of two neighboring swaps,

$$S^1 \wedge \underbrace{S^1 \wedge S^1}_{\text{swap them first}} \quad \text{and} \quad \underbrace{S^1 \wedge S^1}_{\text{swap them next}} \wedge S^1.$$

It suffices for us to show that both operation is multiplication by -1 (and $(-1)^2 = 1$ concludes the proof). Indeed, recall the suspension isomorphism $\pi_2(S^1 \wedge S^1) \to \pi_3(S^1 \wedge (S^1 \wedge S^1))$ given by $\mathrm{id}_{S^1} \wedge \bullet$ fits in the commutative diagram

$$\begin{tikzcd} \pi_2(S^1 \wedge S^1) \arrow[r, "\mathrm{swap}"] \arrow[d, "\cong"'] & \pi_2(S^1 \wedge S^1) \arrow[d, "\cong"] \\ \pi_3(S^1 \wedge (S^1 \wedge S^1)) \arrow[r, "\mathrm{id}_{S^1} \wedge \mathrm{swap}"'] & \pi_3(S^1 \wedge (S^1 \wedge S^1)) \end{tikzcd}$$

Thus, we see that $\mathrm{id}_{S^1} \wedge \mathrm{swap}$ is multiplication by -1. A similar proof shows the other case. ■

More generally, the proof above also adapts to show.

Lemma A.2. Let $\sigma \in \Sigma_n$ be a permutation on n variables act on $S^n = (S^1)^{\wedge n}$. Then the induced map $\pi_n(S^n) = \mathbb{Z} \to \pi_n(S^n) = \mathbb{Z}$ is multiplication by the sign of the permutation σ.

Proof. Observe that $(\mathrm{id}_{S^n})^p \wedge \mathrm{swap} \wedge (\mathrm{id}_{S^n})^q$ for $p + q = n - 2$ also has degree -1 for the same reason as above. Any permutation can be decomposed into a composition of these adjacent swaps (see `BubbleSort` in computer science, for example). This concludes the proof. ■

Remark A.3. Another way to see this could be to note that S^n is the one-point compactification of $\mathbb{R}^n$, and one-point compactification turns Cartesian products to smash products.

B $\mathbb{E}_n$-Structures on Chromatic Spectra

B.1 Morava K-Theory

The purpose of this subsection is to record a cool construction of an $\mathbb{E}_1$-ring structure on the Morava K-theories $K(n)$. The notetaker saw this at **Jeremy Hahn**'s course on the **chromatic splitting conjecture** in Spring 2026. For simplicity we will stick with $p = 2$. Since the original presentation was formulated in the language of ∞-categories, translating it into the framework used in these notes requires a small amount of informality; accordingly, some details in this subsection will be treated heuristically.

Recall MU represents the complex cobordism spectrum. There is a 2-periodic version of MU constructed as follows.

Definition B.1. We define MUP as $\bigvee_{n\in\mathbb{Z}} \Sigma^{2n}\,\mathrm{MU}$. This has the property that

$$\pi_*\,\mathrm{MUP} \cong \mathbb{Z}[x_1, ..., x_n, ...][u^{\pm 1}], |u| = 2.$$

Remark B.2. Strictly speaking, to get the $\mathbb{E}_\infty$ ring structure on MUP, one should define it as the Thom spectrum of the stable complex J-homomorphism $\mathrm{BU}\times\mathbb{Z} \to \mathrm{Pic}(\mathbb{S})$.

Definition B.3. Let $\mathrm{Pic}\,\mathrm{MUP}$ denote the category of $\otimes$-invertible MUP-modules and automorphisms. We have a subcategory

$$\mathrm{BGL}_1\,\mathrm{MUP} \subseteq \mathrm{Pic}\,\mathrm{MUP}\,.$$

where $\mathrm{BGL}_1\,\mathrm{MUP}$ is the component at MUP and its automorphisms. This is an honest space as everything is enriched/tensored over spaces.

The homotopy groups of $\mathrm{BGL}_1\,\mathrm{MUP}$ are given as follows:

Proposition B.4. We have that

$$\pi_*\,\mathrm{BGL}_1\,\mathrm{MUP} \cong \begin{cases} 0, \text{ if } * = 0 \\ \pi_0(\mathrm{MUP})^\times, \text{ if } * = 1 \\ \pi_{*-1}(\mathrm{MUP}), \text{ if } * > 1 \end{cases}.$$

Let us consider any map

$$S^3 \xrightarrow{f} \mathrm{BGL}_1\,\mathrm{MUP}\,.$$

as an element x in

$$\pi_3\,\mathrm{BGL}_1\,\mathrm{MUP} \cong \pi_2\,\mathrm{MUP}\,.$$

Let $\mathrm{Thom}(f)$ denote the **Thom spectra** of this map. If this term sounds foreign, it turns out in this case, one has the equivalence:

$$\mathrm{Thom}(f) = \mathrm{cofiber}(\Sigma^2\,\mathrm{MUP} \xrightarrow{\cdot x} \mathrm{MUP}).$$

Theorem B.5. $\mathrm{Thom}(f) = \mathrm{cofiber}(\Sigma^2\,\mathrm{MUP} \xrightarrow{\cdot x} \mathrm{MUP})$ is an $\mathbb{E}_1$-ring, and in fact an $\mathbb{E}_1$-MUP-algebra.

Proof. We first observe that S^3 is a group-like $\mathbb{E}_1$-space, since S^3 is connected and $S^3 = \Omega\mathbb{HP}^\infty$.

Now we can take
$$B^2 \operatorname{GL}_1 \mathrm{MUP} = B(\mathrm{BGL}_1 \mathrm{MUP}),$$
and f correspons to, using the loop-suspension adjunction, a map we denote using the same symbol
$$\Sigma S^3 = S^4 \xrightarrow{f} B^2 \operatorname{GL}_1 \mathrm{MUP}$$
Consider an extension:
$$\begin{tikzcd} S^4 \arrow[r, "f"'] \arrow[d] & B^2 \operatorname{GL}_1 \mathrm{MUP} \\ \mathbb{HP}^\infty \arrow[ur, dashed] & \end{tikzcd}$$
It turns out obstructions to lifting these are all in the odd homotopy groups of $B^2 \operatorname{GL}_1 \mathrm{MUP}$. Because $\pi_* \mathrm{MUP}$ is concentrated in even degrees, so is $\pi_* B^2 \operatorname{GL}_1 \mathrm{MUP}$. Thus, <u>this is always possible</u>.

Choose a dashed map lifting f and take Ω. Thus, we get that
$$f : S^3 \to \mathrm{BGL}_1 \mathrm{MUP}$$
is the Ω **of some map**. This shows that $\mathrm{Thom}(f) = \mathrm{cofiber}(\Sigma^2 \mathrm{MUP} \xrightarrow{\cdot x} \mathrm{MUP})$ is an $\mathbb{E}_1$-ring and furthermore an $\mathbb{E}_1 - \mathrm{MUP}$-algebra, since we are taking the quotient induced by the loop of some map. ■

Remark B.6. As seen from the proof, the $\mathbb{E}_1$-structure depends on the choice of lift and is highly non-canonical.

Remark B.7. We can also consider any sequence of elements $f_1, f_2, f_3, ... \in \pi_2 \mathrm{MUP}$ and consider
$$S^3 \times S^3 \times S^3 \times ... \to \mathrm{BGL}_1 \mathrm{MUP}$$
to become
$$\Omega(\mathbb{HP}^\infty \times ... \to \mathrm{BGL}_1 \mathrm{MUP}).$$
From here we see that MUP mod any sequence of elements in π_2 is an $\mathbb{E}_1$-MUP-algebra.

Definition B.8. Take the prime $p = 2$, we consider the quotient
$$R = \mathrm{MUP} / (2u, x_1 - u, x_2 u^{-1}, x_3 u^{-2}, ...).$$
This is can be made into an $\mathbb{E}_1$-ring non-canonically. In this case recall that $\pi_* \mathrm{MUP} \cong \mathbb{Z}[x_1, x_2, ...][u^{\pm 1}]$, then
$$\pi_* R \cong \mathbb{F}_2[x_1^{\pm 1}].$$
This has the property that it is a **graded field**, and an $\mathbb{E}_1$-ring where every module is free. This turns out to be exactly the **Morava K-theories** at $p = 2$.

The $K(n)$'s are also $\mathbb{E}_1$ at other primes. In general, one has that:

Theorem B.9. The Morava K-theories $K(n)$ are $\mathbb{E}_1$.

Remark B.10. However, the $K(n)$'s are generally **not** $\mathbb{E}_2$, due to some results on power operations.

B.2 Brown-Petersen Spectrum

We have seen a lot of examples of ring spectra that are either $\mathbb{E}_1$ or $\mathbb{E}_\infty$. Here we discuss on the structures that (truncated) Brown-Petersen spectra have.

In Problem 1 of [May75], Peter May asked:

Question B.11. Does BP have the structure of an $\mathbb{E}_\infty$-ring spectrum?

This remained an open problem for a while, and there were sufficient motivations to suspect that BP is $\mathbb{E}_\infty$. After all, BP was supposed to be the a variant of MU (which is indeed an $\mathbb{E}_\infty$-ring spectrum) and had quite interesting chromatic properties.

However, this turned out to be false. There are various results related to this. To give a somewhat incomplete account:

1. [HKM01] first showed that BP is not $\mathbb{E}_\infty$ with respect to MU.
2. [Law18] then showed that BP is not even $\mathbb{E}_{12}$ at $p = 2$.
3. [Sen24] then showed that BP is not $\mathbb{E}_{2(p^2+2)}$ at odd prime p. [Sen24] also showed the same result for truncated $\mathrm{BP}\langle n\rangle$ at an odd prime p.

There are however lower bounds on the $\mathbb{E}_n$-structure on BP.

1. [BM11] showed that BP is $\mathbb{E}_4$.
2. [HW22] showed that $\mathrm{BP}\langle n\rangle$ is an $\mathbb{E}_3$-MU-algebra.

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