# Motivic Spectral Sequence. § Background, Motivation.

<u>ECHT.</u> : Birational Geometry & Slice Spectral Sequence. (jt w/ Bachmann; Narain - Yakerson.)

$k = \mathbb{C}$

<u>Conjecture</u> [Orlov] $X, Y$ sm proj $\mathbb{C}$-var's.

$$\mathrm{Perf}(X) \simeq \mathrm{Perf}(Y) \quad [\mathrm{D}\text{-equiv.}]$$

$$\Rightarrow M(X)_{\mathbb{Q}} \simeq \underbrace{M(Y)_{\mathbb{Q}}}_{\Omega^\infty H\mathbb{Z}^{\mathbb{Q}} \wedge \Sigma^\infty_+ X}.$$

- Explain this computationally.

- If $X, Y$ D-equiv. then $K(X) \simeq K(Y)$

$$\Rightarrow \begin{array}{ccc} K_0(X)_{\mathbb{Q}} & \xrightarrow{\sim} & K_0(Y)_{\mathbb{Q}} \\ \wr\Big| \mathrm{ch} & & \wr\Big| \mathrm{ch.} \end{array}$$

$$\bigoplus CH^i(X)_{\mathbb{Q}} \xrightarrow{\sim} \bigoplus CH^i(Y)_{\mathbb{Q}}$$

Ungraded iso. !!

**Rmk.**

$$\begin{tikzcd} & X \times Y \arrow[dl, "p_X"'] \arrow[dr, "p_Y"] & \\ X & & Y \end{tikzcd}$$

FM transform

- $\Phi_{\mathcal{E}} = p_{Y*}\left(\mathcal{E} \overset{\mathbb{L}}{\otimes} p_X^*(-)\right) : \mathrm{Perf}(X) \to \mathrm{Perf}(Y)$

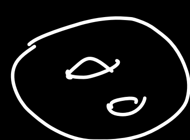
$\mathcal{O}_x \quad x \in X$

$\sim \mathcal{F}$

- Swaps dim 0 cycles $\longleftrightarrow$ codim 0.

$\mathcal{O}_{x} \longmapsto \mathcal{O}_Y \quad [\mathcal{E} = \mathcal{O}_{X\times Y}]$.

**Evidence** 0. dim is a D-inv.

1. Curves 
2. $X$ has $\omega_X$ ample / antiample (Bondal–Orlov)

} D-type $\Updownarrow$ iso type.

3. $X = K3$. Huybrechts. $M(X)_{\mathbb{Q}} \simeq M(Y)_{\mathbb{Q}}$ is not realized by

trivial alg. equiv.

# 4. Kawamata

- $X \overset{\supseteq U \longrightarrow}{\dashleftarrow\dashrightarrow} Y$ bir equiv iff $k(X) \simeq k(Y)$.

$\in \mathrm{Sm}_k$.

$\mathbb{A}^1 \dashleftarrow\dashrightarrow \mathbb{P}^1$

$\mathbb{C}(t)$.

§ Mot/slice S.S.

$$H^{i-j}_{mot}(X; \mathbb{Z}(-j)) \Longrightarrow K_{-i-j}(X).$$

Chow groups

$$\underline{K}^M_n \left\{ \begin{array}{lll} H^{00} & & \\ H^{11} & H^{21} & \\ H^{22} & H^{32} & H^{42} \end{array} \right. = E_2^{i,j}$$

Bloch-Lichtenbaum, Friedlander-Suslin, Levine,

Voevodsky

In light of the mot S.S.

Orlov asked: to what extent does

the $E_\infty$ page of the SS

determine the $E_2$ (rationally) ?

<u>Recall</u> that the mot. s.s. comes from

a tower of mot. Spectra.

$$\cdots \to f_1 E \to f_0 E \to f_{-1} E \to \cdots \to \overset{(KGL)}{\widetilde{E}}$$

- $SH^{eff}(k) = \langle \Sigma^\infty_+ X \rangle \hookrightarrow SH(k).$

eff. mot. sp

no negative $\mathbb{G}_m$'s allowed.

- $\Sigma^n_{\mathbb{G}_m} SH^{eff}(k) =: SH^{eff}(k)(n) \qquad n \in \mathbb{Z}$

$$\begin{tikzcd} SH^{eff}(k)(n) \arrow[d, shift left, "i_n"] \\ SH(k) \arrow[u, bend left, "r_n"] \end{tikzcd}$$

$f_n E := i_n r_n E.$

- to constr. the mot. s.s. = SSS for KGL.

$s_n E := \mathrm{cof}(f_{n+1} E \to f_n E).$ (slices.)

. by "mot. Bott periodicity"

$$s_n KGL \simeq \underset{\substack{\shortparallel \\ (S^1 \wedge \mathbb{G}_m)^{\wedge n}}}{\Sigma^{2n,n}} s_0 KGL.$$

$$E_2^{p,q} = (s_0 KGL)^{p-q,-q}(X) \Rightarrow K_{-p-q}(X).$$

. We then want to prove that

$$s_0(KGL) \simeq H\mathbb{Z}$$

spectrum rep motivic coh.

$$\Uparrow$$

$$\boxed{s_0(\mathbb{1}) \simeq H\mathbb{Z}}$$

<u>Thm</u> (Voevodsky, Levine) $s_0(\mathbb{1}) \simeq H\mathbb{Z}$ over any field $k$.

New proof (jt w/ T. Bachmann)

Based on mot. infinite loop space thy.
(E. - Hoyois - Khan - Sosnilo - Yakerson)
(Ananyevskiy - Druzhinin - Latushin - Neshitov - Panin)

framed ...

$$\begin{tikzcd} \underbrace{\mathcal{H}^{fr}(k)}_{\text{motivic spaces.}} \arrow[d, "\Sigma^\infty_{fr}"'] & \{\text{gp like } \mathbb{E}_\infty\text{-spaces.}\} \arrow[l] \arrow[d, "\Sigma^\infty"] \\ SH(k) & Sp. \end{tikzcd}$$

Im is "very eff mot sp."

Im is conn sp.

Objects here are: $\mathcal{F} : Sm_k^{op} \longrightarrow Spc$

- Nis sheaf.
- $\mathbb{A}^1$-inv.
- $W \xrightarrow{f} X$ $f$ is finite syntomic $f$ flat. $\mathbb{L}_f$ is tor am. $[0,1]$,

  $+\ \underline{[\mathbb{L}_{W/X}] \overset{\alpha}{\simeq} 0}$ in $\underline{K(Z)}$. $\Omega^\infty K(Z)$

  pushfwd map. $(f,\alpha)_*$.

- $+$ coherences.

- $\mathcal{F}$ has the str. of a Comm mon obj in presheaves, $\mathbb{E}_\infty$

  ask that $\pi_0\mathcal{F}$ is gplike.

One thing that you might like ab $\mathcal{H}^{fr}$.

there is an actual sheaf.

- $\cdots \to f_2 E \to f_1 E \to f_0 E = E$

Ask: how does this tw look like in $\mathcal{H}^{fr}$

- Let's understand the following piece of the tw:

$$f_0 \mathcal{E} \to \mathcal{E} \to \mathrm{cof}(f_0\mathcal{E} \to \mathcal{E}) = s_0 \mathcal{E}$$

e.g. $X \in Sm_k$, $U \xhookrightarrow[\text{open immersion}]{} X$

$(\mathbb{A}^1 \hookrightarrow \mathbb{P}^1)$

Observe that we can take the cofiber in mot. spaces

$$U \hookrightarrow X \to X/U \simeq Th(V_Z) \quad Z \hookrightarrow X \text{ comp of } U$$

$$\cong \underbrace{S^{2c,c}}_{(\mathbb{P}^1)^{\wedge c}} \wedge Z_+ \qquad V_Z \cong \mathcal{O}^{\oplus c}$$

Now as soon as $c \geq 1$

notice that $\Sigma^\infty(S^{2c,c} \wedge Z_+ \simeq X/U)$

$\in SH^{eff}(1)$

$$\Rightarrow \Sigma^\infty U^{fr} \underset{s_0}{\simeq} \Sigma^\infty X^{fr}$$

free framed mot. spa on $X, U$.

- $Sm_k \xrightarrow{(\ )^{fr}} \mathcal{H}^{fr}(k)$

**Def** $U \overset{j}{\hookrightarrow} X$ open imm in Smk.

$j$ is said to be birational if

$j$ contains all the gen. pt's of $X$.

$$L^0_{bir} : \mathcal{H}^{fr}(k) \longrightarrow \mathcal{H}^{fr}(k)$$

inverts all $U^{fr} \xrightarrow{j^{fr}} X^{fr}$

whenever $j$ is birational

By above discussion $\Sigma^\infty_{fr} L^0_{bir}\text{-equiv} \subseteq S_0\text{-equiv}$.

**Prop** $X \in \mathcal{H}^{fr}(k)$ then

$$\Sigma^\infty_{fr} X \longrightarrow \Sigma^\infty_{fr} L^0_{bir} X$$
$$\cong$$
$$S_0(\Sigma^\infty_{fr} X).$$

More generally $U \hookrightarrow X$ is a $d$-bir $\left(\begin{smallmatrix} 0\text{-bir} \\ = \text{bir} \end{smallmatrix}\right)$

equiv if $x \in X$ $\mathrm{cod}_X(x) \leq d$

then $x \in U$.

**Prop** $\Sigma^\infty_{fr} L^d_{bir} \simeq f^d$.

Upshot: given a geometric interpretation of these slices.

To compute the zero slice of the sphere.

$$\begin{tikzcd}
\Sigma^\infty_{fr} \underline{Pt}^{fr} \arrow[r, "\text{forget framing}"'] \arrow[rr, bend left, "\deg L_{bm}"] & \Sigma^\infty_{fr} F\underline{flat} \arrow[r, "\deg."'] \arrow[d, phantom, "\wr"] & \underline{\Sigma^\infty_{fr} \mathbb{Z}} \arrow[d, "\cong"] \\
& \{R \to T\}_{\text{flat}} \arrow[d, "\cong"] & H\mathbb{Z} \\
& \Sigma^\infty_{fr} Gr \times \mathbb{Z} \arrow[uur, "s_0^{eff}"] &
\end{tikzcd}$$

constant sheaf $\mathbb{Z}$ (Hoyois).

$\in SH(k)$

Hoyois, Spitzweck, Levine

mot. coh.

Hoyois - Jelisiejew - Nardin - Tonini - Yakerson

$\infty$-grassmannian

reps K-thy

$s_0 KGL = H\mathbb{Z}$

$\operatorname{Spec} R = S$

$= \{$ a finite syntomic algebra $R \to T$ + $\alpha \}$.

Observe that $Gr \underset{\mathbb{A}^1}{\simeq} pt.$

Nardin - Yakerson. char $k = 0$.

$\alpha \in B_r(X) \rightsquigarrow K^\alpha(X) = K(\mathrm{Perf}^\alpha(X))$.

- $H^{p-q}(X; \mathbb{Z}^\alpha(-q)) \Rightarrow K^\alpha_{-p-q}(X)$. twist
- improves prior results of Lieblich - Kahn.