FREE LOOP SPACES AND TOPOLOGICAL COHOCHSCHILD HOMOLOGY

Joint with: Anna Marie Bohmann and Brooke Shipley

TODAY: Describe a new approach to homology of free loop spaces

$$H_*(\mathcal{L}X;k), \quad \mathcal{L}X = \mathrm{Map}(S^1, X)$$

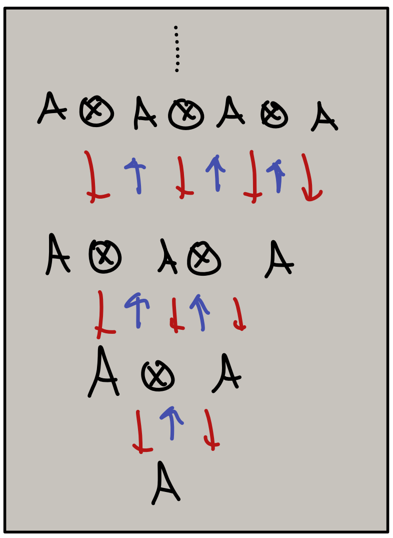
– via topological coHochschild homology

Thm [Gromoll–Meyer 1969]: For M a simply connected closed smooth manifold, if for some field k the Betti numbers $\dim H_i(\mathcal{L}M;k)$ are unbounded, then M has infinitely many distinct closed geodesics in any Riemannian metric.

Hochschild Homology:

Let k be a commutative ring, and A a k-algebra. Let $HH(A)_\bullet$ denote the simplicial k-module

$$HH(A)_r = A^{\otimes_k r+1}$$



Face maps: $d_i : A^{\otimes r+1} \longrightarrow A^{\otimes r}$

$$d_i(a_0 \otimes a_1 \otimes \cdots \otimes a_r) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_r & 0 \le i < r \\ a_r a_0 \otimes \cdots \otimes a_{r-1} & i = r \end{cases}$$

$$d_i = \begin{cases} \mathrm{Id}^{i} \otimes \phi \otimes \mathrm{Id}^{r-i-1} & 0 \le i < r \\ (\phi \otimes \mathrm{Id}^{r-1}) \circ \tau \end{cases}$$

Degeneracy maps: $s_i : A^{\otimes r} \longrightarrow A^{\otimes r+1}$

$$s_i = \mathrm{Id}^{i+1} \otimes \eta \otimes \mathrm{Id}^{r-i-1} \qquad 0 \le i < r$$

- There is an associated chain complex $C_\bullet(A)$ with $d = \Sigma(-1)^i d_i$

Def: $HH_q(A) = H_q(C_\bullet(A)) = \pi_q(|HH(A)_\bullet|)$

Algebras:

Commutative ring k

Algebra A over k

Product:

$\phi : A \otimes_k A \longrightarrow A$

Unit:

$\eta : k \longrightarrow A$

Associativity:

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "id \otimes \phi"] \arrow[d, "\phi \otimes id"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "\phi"] \\ A \otimes A \arrow[r, "\phi"] & A \end{tikzcd}$$

Unitality:

$\phi \circ (id \otimes \eta) = id = \phi \circ (\eta \otimes id)$

Coalgebras:

Commutative ring k

Coalgebra C over k

Coproduct:

$\Delta : C \longrightarrow C \otimes_k C$

Counit:

$\epsilon : C \longrightarrow k$

Coassociativity:

$$\begin{tikzcd} C \otimes C \otimes C \arrow[dr, phantom, "\circlearrowright"] & C \otimes C \arrow[l, "id \otimes \Delta"'] \\ C \otimes C \arrow[u, "\Delta \otimes id"] & C \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd}$$

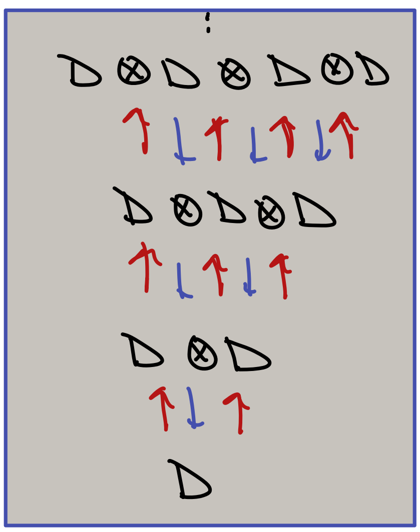
Counitality:

$(id \otimes \epsilon) \circ \Delta = id = (\epsilon \otimes id) \circ \Delta$

coHochschild Homology [Doi 70's]

Let D be a cocommutative coalgebra over a field k. Define a cosimplicial k-module $\mathrm{coHH}^\bullet(D)$, $\mathrm{coHH}^r(D) = D^{\otimes_k r+1}$

Coface $\delta_i : D^{\otimes(r+1)} \longrightarrow D^{\otimes(r+2)}$



$$\delta_i = \begin{cases} \mathrm{Id}^{i} \otimes \Delta \otimes \mathrm{Id}^{(r-i)} & 0 \le i \le r \\ \tau \circ (\Delta \otimes \mathrm{Id}^r) & i = r+1 \end{cases}$$

τ cycles first entry to last

Codegeneracies:

$\sigma_i : D^{\otimes(r+2)} \longrightarrow D^{\otimes(r+1)}$

$\sigma_i = \mathrm{Id}^{(i+1)} \wedge \epsilon \wedge \mathrm{Id}^{(r-i)} \qquad 0 \le i \le r$

- There is an associated cochain complex $C^*(D)$ with $\delta = \Sigma (-1)^i \delta_i$

Def: $\mathrm{coHH}_q(D) = H^q(C^*(D))$

Topological Hochschild Homology [Bökstedt]

Algebra		Topology
Ring A	$\rightsquigarrow$	Ring spectrum R
$\otimes$	$\rightsquigarrow$	$\wedge$
$HH(A)_\bullet$	$\rightsquigarrow$	$THH(R)_\bullet$

$\phi : R \wedge R \longrightarrow R$

$\eta : \mathbb{S} \longrightarrow R$

$$\vdots$$
$$R \wedge R \wedge R \wedge R$$
$$\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow$$
$$R \wedge R \wedge R$$
$$\downarrow\uparrow\downarrow\uparrow\downarrow$$
$$R \wedge R$$
$$\downarrow\uparrow\downarrow$$
$$R$$

$$\boxed{THH(R) = |THH(R)_\bullet|}$$

$$\begin{array}{ccc}
\text{Hochschild homology} & \rightsquigarrow & \text{Topological Hochschild Homology} \\
\Big\downarrow & & \Big\downarrow \\
\text{CoHochschild homology [Doi]} & \rightsquigarrow & \text{Topological coHochschild homology [Hess-Shipley]}
\end{array}$$

Topological CoHochschild Homology [Hess-Shipley]

Algebra		Topology
Coalgebra D	$\rightsquigarrow$	coalgebra in spectra C
$\otimes$	$\rightsquigarrow$	$\wedge$
$\mathrm{CoHH}^\bullet(D)$	$\rightsquigarrow$	$\mathrm{CoTHH}^\bullet(C)$

$\Delta : C \to C \wedge C$

$\epsilon : C \to \mathbb{S}$

$$\begin{array}{c}
\vdots \\
\uparrow\ \downarrow\uparrow\downarrow\uparrow\downarrow\uparrow \\
C \wedge C \wedge C \\
\uparrow\downarrow\uparrow\downarrow\uparrow \\
C \wedge C \\
\uparrow\downarrow\uparrow \\
C
\end{array}$$

$$\mathrm{coTHH}(C) = \mathrm{Tot}(\mathrm{coTHH}^\bullet(C))$$

Examples of Coalgebras in Spectra:

For a space X, the suspension spectrum
$\Sigma^\infty_+ X$ is a coalgebra.

$$\Delta : X \longrightarrow X \times X$$

yields comultiplication

$$\Sigma^\infty_+ X \to \Sigma^\infty_+ X \wedge \Sigma^\infty_+ X$$

Thm [Hess-Shipley, Kuhn, Malkiewich]: For X a simply connected space,

$$\mathrm{coTHH}(\Sigma^\infty_+ X) \simeq \Sigma^\infty_+ \mathcal{L} X$$

Q: How do we compute the homology of $\mathrm{coTHH}(C)$?

Thm [Bohmann-G-Høgenhaven-Shipley-Ziegenhagen]:

Let k be a field, C a coalgebra spectrum. There is a "coBökstedt" spectral sequence for calculating $H_*(\mathrm{coTHH}(C);k)$ with E_2-term

$$\underbrace{E_2^{*,*} = \mathrm{coHH}_*(H_*(C;k))}_{\text{algebraic theory}}$$

Bökstedt spectral sequence:

$$E^2_{*,*} = HH_*(H_*(R;k)) \Rightarrow H_*(\mathrm{THH}(R);k)$$

Look at case $C = \Sigma^\infty_+ X$

Corollary: For X simply connected. If for each s there is an r such that $E_r^{s,s+i} = E_\infty^{s,s+i}$,

$$E_2^{*,*} = \mathrm{CoHH}_*(H_*(X;k)) \Rightarrow H_*(\mathcal{L}X;k)$$

Goal: Understand algebraic structure in the coBökstedt spectral sequence.

Prop: [Angeltveit-Rognes]: If A is commutative, $HH_*(A)$ is flat as an A-module, $HH_*(A)$ is an A-Hopf algebra.

Recall: For a commutative ring A, and A-Hopf algebra is an A-module with:

- $\phi: H \otimes_A H \longrightarrow H$ product ⟵ associative and unital
- $\eta: A \longrightarrow H$ unit
- $\psi: H \longrightarrow H \otimes_A H$ coproduct ⟵ coassociative and counital
- $\epsilon: H \longrightarrow A$ counit
- $\chi: H \longrightarrow H$ Antipode

satisfying various compatibilities...

... e.g. :

$$\begin{tikzcd}
H \otimes_A H \arrow[r, "\phi"] \arrow[d, "\psi\otimes\psi"'] & H \arrow[r, "\psi"] & H \otimes_A H \\
H \otimes_A H \otimes_A H \otimes_A H \arrow[rr, "id\otimes\gamma\otimes id"] & & H \otimes_A H \otimes_A H \otimes_A H \arrow[u, "\phi\otimes\phi"']
\end{tikzcd}$$

Q: Why is $HH_*(A)$ an A-Hopf algebra?

Idea: For commutative A, identify $HH(A)_\bullet$ as simplicial tensor

$$HH(A)_\bullet = A \otimes S^1_\bullet$$

<u>simplicial map</u> $\rightsquigarrow$ <u>map on $HH_*(A)$</u>

$\nabla: S^1_\bullet \vee S^1_\bullet \to S^1_\bullet$ (fold) $\rightsquigarrow$ $\phi: HH_*(A) \otimes_A HH_*(A) \to HH_*(A)$

$\eta: * \to S^1_\bullet$ $\rightsquigarrow$ $\eta: A \to HH_*(A)$

$\epsilon: S^1_\bullet \to *$ $\rightsquigarrow$ $\epsilon: HH_*(A) \to A$

$\psi: dS^1_\bullet \to S^1_\bullet \vee S^1_\bullet$ (pinch) $\rightsquigarrow$ $\psi: HH_*(A) \to HH_*(A) \otimes_A HH_*(A)$

$\chi : dS^1_\bullet \xrightarrow[\text{flip}]{} dS^1_\bullet \quad \rightsquigarrow \quad \chi : HH_*(A) \to HH_*(A)$

Thm: [Angeltveit–Rognes]: Let R be a commutative ring spectrum. In the Bökstedt spectral sequence for $H_*(THH(R);k)$ if each term $E^r_{*,*}$ for $r \geq 2$ is flat over $H_*(R;k)$, then it is a spectral sequence of Hopf algebras over $H_*(R;k)$.

i.e. has a product and a coproduct satisfying

Leibniz rule: $d(xy) = d(x)y \pm x\,d(y)$

coLeibniz rule: $\psi \circ d = (d \otimes 1 \pm 1 \otimes d) \circ \psi$

Hope: we would like similar structure in coBökstedt spectral sequence.

$\rightsquigarrow$ algebraic structure over $\underbrace{H_*(C;k)}_{\text{coalgebra}}$

Def: For C a coalgebra over a field k, a right C-comodule M is a k module with linear coaction map

$\rho_M : M \longrightarrow M \otimes C$

that is coassociative and counital.

Def: For M a right C-comodule and N left C-comodule the cotensor product $M \square_C N$ is the equalizer:

$$M \square_C N \to M \otimes N \underset{id \otimes \rho_N}{\overset{\rho_M \otimes id}{\rightrightarrows}} M \otimes C \otimes N$$

Def: Let C be a cocommutative coalgebra over a field k. A $\square_C$-Hopf algebra is a C bicomodule H together with maps of bicomodules:

- $\Delta : H \longrightarrow H \square_C H$ coproduct
- $\epsilon : H \longrightarrow C$ counit

coassociative and counital

- $\mu : H \square_C H \longrightarrow H$ product
- $\eta : C \longrightarrow H$ unit

associative and unital

- $\chi : H \longrightarrow H$ antipode

satisfying various compatibilities...

Def: A right C-comodule M is coflat if $M \square_C -$ is an exact functor.

Thm [Bohmann-G-Shipley]: For X a simply connected space and k a field if $H_*(\Omega X;k)$ is coflat as a comodule over $H_*(X;k)$, then $H_*(\Omega X;k)$ is a $\square_{H_*(X;k)}$-Hopf algebra.

Thm: [Bohmann-G-Shipley]: Let C be a connected cocommutative coalgebra in spectra. If for each $r \geq 2$ $E_r^{*,*}$ is coflat over $H_*(C;k)$ then the coBökstedt spectral sequence is a spectral sequence of $\square_{H_*(C;k)}$-Hopf algebras.

Goal: compute $H_*(\Omega X;k)$.

[Kuribayashi + Yamaguchi, 1997]:

For X simply connected space with

$$H^*(X;\mathbb{Z}/p) = \Lambda_{\mathbb{Z}/p}(x_{i_1}, x_{i_2}) \quad |x_i| = i \text{ odd}$$

where $i_1 \leq i_2 \leq 2i_1 - 2$ and $p \geq 3$, they computed $H_*(\Omega X;\mathbb{Z}/p)$:

Thm [Bohmann-G-Shipley]: let k be a field of characteristic p and X a simply connected space whose cohomology is:

$$H^*(X;k) = \Lambda_k(x_{i_1}, x_{i_2}, \ldots x_{i_n})$$

$|x_i| = i$ odd, $i_j \leq i_{j+1}$.

When $\dfrac{i_n + \sum_{j=1}^{n} i_j}{i_1 - 1} \leq p$,

$$H_*(\Omega X;k) \cong \Lambda_k(y_{i_1}, \ldots y_{i_n}) \otimes k\Gamma[w_{i_1}, \ldots w_{i_n}]$$

as a graded module, where

$|y_i| = i$, $|w_i| = i-1$.

For appropriate choices of p, recover some known calculations of the homology of free loop spaces for:

SU(n), Sp(n) ...

Prop [Bohmann - G. - Shipley]: under certain coflatness conditions,

$$E_2^{**} = \mathrm{CoHH}_*(H_*(C;k))$$

is a $\square_{H_*(C;k)}$-bialgebra, and the shortest nonzero differential in lowest total degree maps from a $\square_{H_*(C;k)}$-algebra indecomposable to a $\square_{H_*(C;k)}$-coalgebra primitive.