

A shadow perspective on equivariant Hochschild homology



eCHT

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Joint work with

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Twisted THH of C_n -spectra

C_n -twisted THH

[Angeltveit, Blumberg, Gerhardt, Hill, Lawson, Mandell]
2018

Let $\mathcal{R}$ be a C_n -ring spectrum (orthogonal)

Let M be an $\mathcal{R}$ -bimodule.

C_n -twisted THH

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Let R be a C_n -ring spectrum (orthogonal)

Let M be an R -bimodule.

$\Rightarrow$ simplicial spectrum $B_{\bullet}^{cyc, C_n}(R; M)$

with $B_g^{cyc, C_n}(R; M) = M \wedge R^{\wedge g}$

C_n -twisted THH

Simplicial spectrum $B_{\bullet}^{cyc, C_n}(\mathbb{R}; M)$

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- 0th face given by right $\mathbb{R}$ -action on M

C_n -twisted THH

Simplicial spectrum $B_{\bullet}^{cyc, C_n}(\mathcal{R}; M)$

with $B_q^{cyc, C_n}(\mathcal{R}; M) = M \wedge \mathcal{R}^{\wedge q}$.

- 0th face given by right $\mathcal{R}$ -action on M
- i th face ($0 < i < q$) given by mult in $\mathcal{R}$

C_n -twisted THH

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- 0th face given by right $\mathbb{R}$ -action on M
- i th face ($0 < i < q$) given by mult in $\mathbb{R}$
- q th face given by left $\mathbb{R}$ -action on M ,
twisted by C_n -action on $\mathbb{R}$

C_n -twisted THH

$$\mathrm{THH}_{C_n}(\mathbb{R}; M) = |\mathcal{B}^{\mathrm{cyc}, C_n}(\mathbb{R}; M)|$$

– Model of $N_{C_n}^{S^1}(\mathbb{R})$ (the relative norm)
when $M = \mathbb{R}$.

C_n -twisted THH

Natural C_n -spectrum structure
in general and S^1 -spectrum
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How to compute it?

Equivariant Bökstedt SS [AGHKK]

Let $G \leq S^1$ be a finite subgroup with generator g .

Let $\mathcal{R}$ be an orthogonal G -ring spectrum.

Let E be a commutative G -ring spectrum on which g acts trivially.

Equivariant Bökstedt SS [AGHKK]

If $E_{\star}(R)$ is flat over $E_{\star}$, $\exists$ strongly converging SS of Mackey functors

$$E_{s, \star}^2 = \varinjlim_s H H_{E_{\star, G}}^{E_{\star, G}}(E_{\star}(R)) \Rightarrow E_{s+\star}(\iota_{\star}^* THH_{\star}(R)).$$

Here: $E_{\star}(R) = \pi_{\star}(E \wedge R)$, an $RO(G)$ -graded commutative Green functor

Equivariant Bökstedt SS [AGHKK]

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$$E_{s, \star}^2 = \varinjlim_s \text{HH}_{E_{\star, G}}^{E_{\star, G}}(E_{\star}(R)) \Rightarrow E_{s+\star}(L_G^* \text{THH}_G(R)).$$

↑ Generalization of [BGHL] theory of HH of Green functors to $E_{\star}$ -algebras

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- Evaluated at $G|e$, recover classical Bökstedt SS

$$\text{HH}_s^{E^*}(E^*R; {}^g E^*R) \Rightarrow E_{s+*}(\text{THH}(R; {}^g R))$$

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- R commutative $\Rightarrow$ SS of $RO(G)$ -graded $E_{\star}$ -algebras

Sample computation

$$H_{-}^{C_2} (V_{C_2}^* \otimes HH_{C_2}(MU_{\mathbb{R}}); \mathbb{F}_2)$$

$$\cong \underline{HF}_{2*} [b_1, b_2, \dots] \otimes_{\mathbb{F}_2} \wedge_{\mathbb{F}_2} (z_1, z_2, \dots)$$

as $\underline{HF}_{2*}$ - modules where

$$|b_i| = i\rho \quad \text{and} \quad |z_i| = 1 + i\rho$$

where ρ is the regular rep. of C_2 .

Question

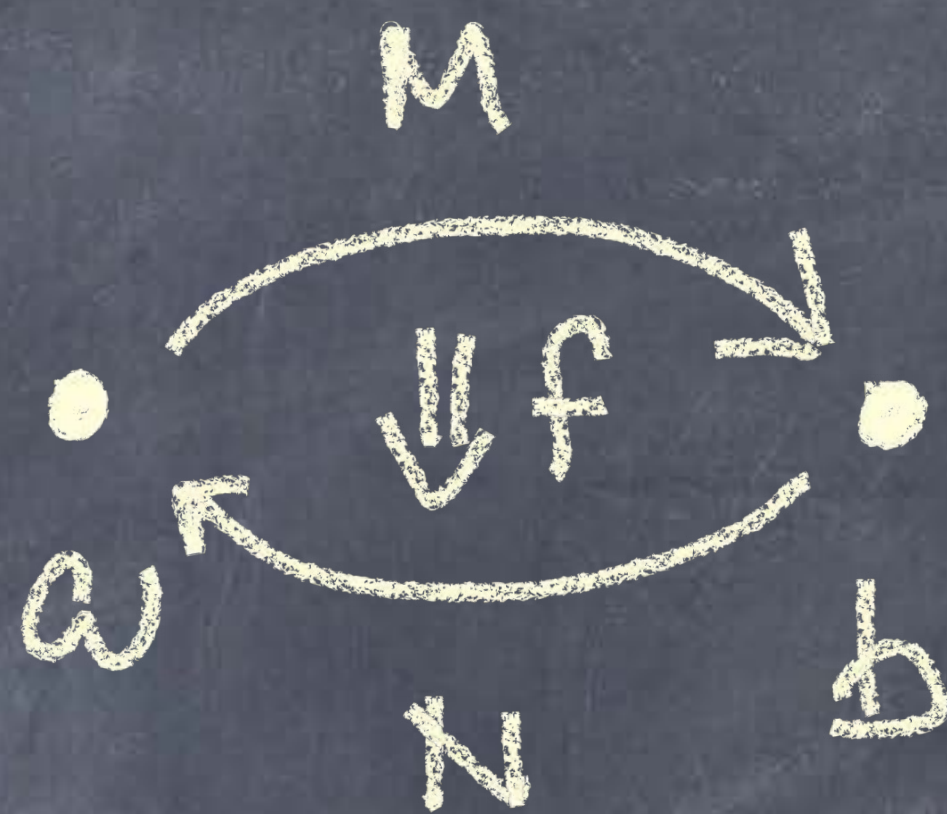
What usual properties of Hochschild
homology (e.g., Morita invariance,
agreement, ...) hold for twisted

THH?

Shadows on bicategories

Bicategories $\mathcal{B}$ specified by:

- $\mathcal{B}_0 = 0\text{-cells (objects)}$
- $\mathcal{B}_1 = 1\text{-cells (morphisms)}$
- $\mathcal{B}_2 = 2\text{-cells}$



Bicategories

$\mathcal{B}$ specified by:

- $a, b \in \mathcal{B}_0 \Rightarrow$ category $\mathcal{B}(a, b)$ w/
$$\begin{cases} \text{Ob } \mathcal{B}(a, b) \subseteq \mathcal{B}_1 \\ \text{Mor } \mathcal{B}(a, b) \subseteq \mathcal{B}_2 \end{cases}$$

Bicategories $\mathcal{B}$ specified by:

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- $a, b, c \in \mathcal{B}_0 \Rightarrow$ functor

$$- \circ - : \mathcal{B}(a, b) \times \mathcal{B}(b, c) \rightarrow \mathcal{B}(a, c)$$

- appropriately associative and unital
wrt distinguished unit objects $u_a \in \mathcal{B}(a, a)$

Example: Ring

$$\text{Ring}_0 = \{\text{associative, unital rings}\}$$

$$\text{Ring}(A, B) = \text{category of } (A, B)\text{-bimodules}$$

$$\mathcal{U}_A = A^A_A$$

$${}_A M_B \odot_B N_C = M \otimes_B N$$

Example : Ring_R , for $(V, \otimes, I)$ nice enough
monoidal model category

Example: Ring_V , for $(V, \otimes, I)$ nice enough
monoidal model category

$$(\text{Ring}_V)_0 = \{A \in \text{Mon}_V \mid A \text{ underlying cofibrant}\}$$

$$\text{Ring}_V(A, B) = \text{Ho}(A \text{ Mod } B)$$