<u>Example</u>: $\mathcal{R}ing_{\mathcal{V}}$, for $(\mathcal{V}, \otimes, \mathbb{I})$ nice enough monoidal model category

$$(\mathcal{R}ing_{\mathcal{V}})_0 = \{A \in Mon_{\mathcal{V}} \mid A \text{ underlying cofibrant}\}$$

$$\mathcal{R}ing_{\mathcal{V}}(A,B) = Ho({}_A Mod_B)$$

$$\mathcal{U}_A = {}_A A_A$$

$${}_A M_B \odot {}_B N_C = |Bar_\bullet(M;B;N)|$$

derived tensor product

## Duality $\mathcal{B}$ bicategory

A dual pair for $a, b \in \mathcal{B}_0$ consists of

$$\begin{tikzcd} a \arrow[r, bend left, "{}_aM_b"] & b \arrow[l, bend left, "{}_bN_a"] \end{tikzcd}$$

# Duality   $\mathcal{B}$ bicategory

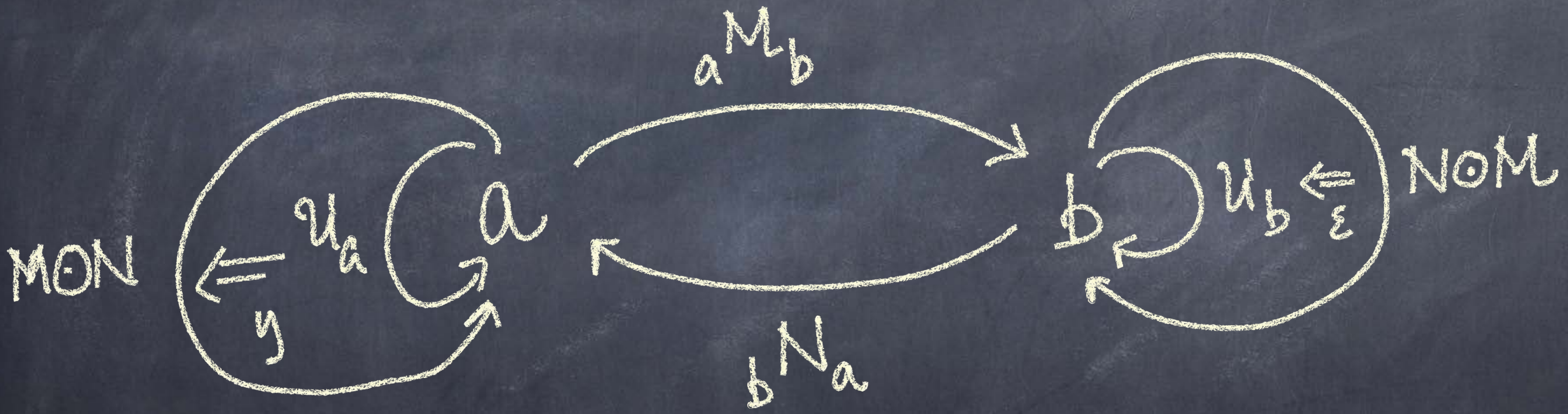
A **dual pair** for $a, b \in \mathcal{B}_0$ consists of

$$\begin{tikzcd}[column sep=huge]
a \arrow[loop, in=120, out=240, looseness=6, "U_a"'] \arrow[loop, in=100, out=260, looseness=14, "MON"'] \arrow[r, bend left, "{}_aM_b"] & b \arrow[l, bend left, "{}_bN_a"] \arrow[loop, in=60, out=300, looseness=6, "U_b"] \arrow[loop, in=80, out=280, looseness=14, "NOM"]
\end{tikzcd}$$

$$U_a \overset{\eta}{\Rightarrow} MON \qquad U_b \overset{\varepsilon}{\Leftarrow} NOM$$

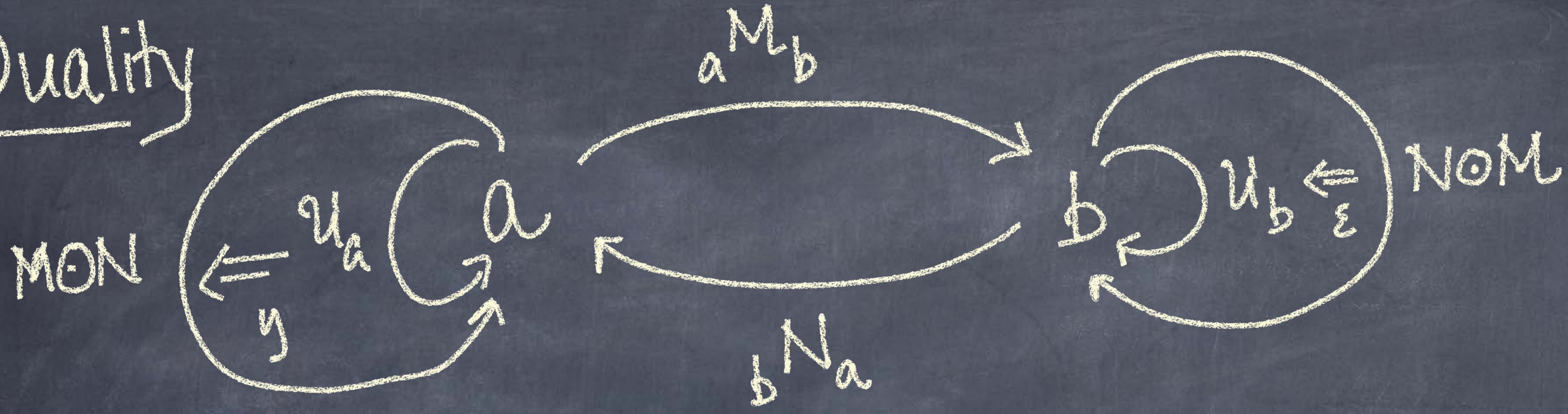
## Duality $\mathcal{B}$ bicategory

A **dual pair** for $a, b \in \mathcal{B}_0$ consists of




satisfying triangle identities, i.e.,

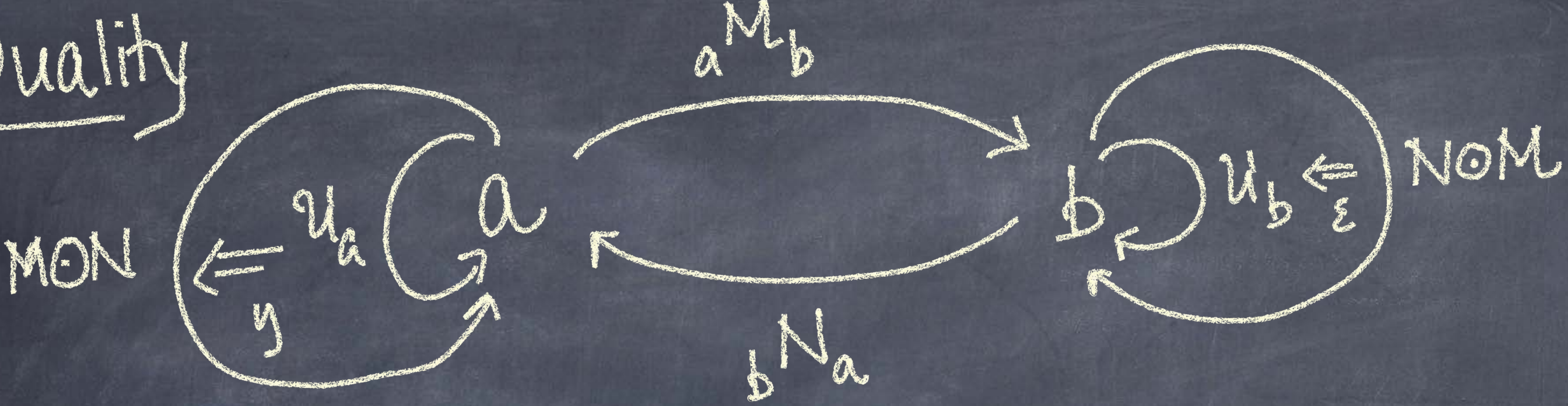
# Duality




$$\begin{tikzcd}
N \arrow[r, "N\circ\eta"] \arrow[dr, "\mathrm{Id}_N"'] & NO(MON) \cong (NOM)ON \arrow[d, "\varepsilon ON"] \\
& N
\end{tikzcd}$$

and similarly for $M$.

# Duality




A dual pair $({}_aM_b, {}_bN_a, \varepsilon, \eta)$ is a **Morita equivalence** if $\varepsilon, \eta$ isomorphisms and $(N, M, \eta^{-1}, \varepsilon^{-1})$ also dual pair.

# Example

$({}_A M_B, {}_B N_A, \varepsilon, \eta)$ dual pair in $\mathcal{R}$ing

$\Rightarrow$ adjunction

$$\begin{tikzcd} \mathrm{Mod}_A \arrow[r, shift left=2, "- \otimes_A M"] \arrow[r, phantom, "\perp"] & \mathrm{Mod}_B \arrow[l, shift left=2, "- \otimes_B N"] \end{tikzcd}$$

## Example

$({}_AM_B, {}_BN_A, \varepsilon, \eta)$ Morita equivalence in Ring

$\Rightarrow$ equivalence

$$\mathrm{Mod}_A \underset{-\otimes_B N}{\overset{-\otimes_A M}{\rightleftarrows}} \mathrm{Mod}_B$$

# Example

$({}_A M_B, {}_B N_A, \varepsilon, \eta)$ Morita equivalence in Ring

$\Rightarrow$ equivalence

$$\begin{tikzcd} \mathrm{Mod}_A \arrow[r, shift left, "- \otimes_A M"] & \mathrm{Mod}_B \arrow[l, shift left, "- \otimes_B N"] \end{tikzcd}$$

$\boxed{\exists \text{ analogous result in } \mathrm{Ring}_W \text{ for } \mathrm{Ho}(\mathrm{Mod}_A), \mathrm{Ho}(\mathrm{Mod}_B).}$

# <u>Shadows</u> [Ponto, 2010], [Ponto-Campbell, 2019]

$\mathcal{B}$ bicategory, $\underline{C}$ category

A $\underline{C}$-shadow on $\mathcal{B}$ consists of:

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- nat'l isos

$$\theta_{a,b} : \langle\!\langle {}_aM_b \odot {}_bN_a \rangle\!\rangle_a \xrightarrow{\cong} \langle\!\langle {}_bN_a \odot {}_aM_b \rangle\!\rangle_b$$

Trace-like behavior! $\quad \forall a,b \in \mathcal{B}_0$

# Shadows [Ponto, 2010], [Ponto-Campbell, 2019]

$\mathcal{B}$ bicategory, $\underline{C}$ category

A $\underline{C}$-shadow on $\mathcal{B}$ consists of:

- functors $\sigma_a : \mathcal{B}(a,a) \longrightarrow \underline{C}$ $\forall a \in \mathcal{B}_0$
- nat'l isos

$$\theta_{a,b} : \sigma_a({}_aM_b \odot {}_bN_a) \xrightarrow{\cong} \sigma_b({}_bN_a \odot {}_aM_b)$$

$$\forall a,b \in \mathcal{B}_0$$

\+ AXIOMS!

# Duality and shadows

$({}_aM_b, {}_bN_a, \varepsilon, \eta)$ dual pair

$\Rightarrow$

$$\begin{tikzcd}
\sigma_a(\mathcal{U}_a) \arrow[r, "\sigma_a(\eta)"] \arrow[dr, "\chi_M"'] & \sigma_a(M \odot N) \cong \sigma_b(N \odot M) \arrow[d, "\sigma_b(\varepsilon)"] \\
& \sigma_b(\mathcal{U}_b)
\end{tikzcd}$$

# Duality and shadows

$({}_aM_b, {}_bN_a, \varepsilon, \eta)$ dual pair

$$\Rightarrow \quad \begin{tikzcd} \tau_a(\mathcal{U}_a) \arrow[r, "\tau_a(\eta)"] \arrow[dr, "\chi_M"'] & \tau_a(M \odot N) \cong \tau_b(N \odot M) \arrow[d, "\tau_b(\varepsilon)"] \\ & \tau_b(\mathcal{U}_b) \end{tikzcd}$$

Rmk: If Morita equivalence, then

$$\chi_M : \tau_a(\mathcal{U}_a) \xrightarrow{\cong} \tau_b(\mathcal{U}_a).$$

Morita invariance of shadows

# Hochschild shadows

## $\mathcal{V}$HH

$(\mathcal{V}, \otimes, \mathbb{I})$ as before

$A \in (\mathrm{Ring}_{\mathcal{V}})_0$ , $M \in {}_A\mathrm{Mod}_A$

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$\mathrm{Bar}_n^{cyc}(A;M) = A^{\otimes n} \otimes M$

$$HH_{\mathcal{V}}(A;M) = |\mathrm{Bar}_\bullet^{cyc}(A;M)| \in \mathcal{V}$$

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$$HH_{\mathcal{V}}(A;M) = |\mathrm{Bar}_\bullet^{cyc}(A;M)| \in \mathcal{V}$$

$$HH_{\mathcal{V}}(A) = HH_{\mathcal{V}}(A;A)$$

# $\mathcal{V}$HH is a shadow

**Proposition:** The set of functors

$$\{ HH_{\mathcal{V}}(A;-) : \mathcal{R}ing_{\mathcal{V}}(A,A) \longrightarrow Ho\mathcal{V} \mid A \in (\mathcal{R}ing_{\mathcal{V}})_0 \}$$

can be equipped with the structure of a $Ho\mathcal{V}$-shadow on $\mathcal{R}ing_{\mathcal{V}}$.

# $\mathcal{V}HH$ is a shadow

**Proposition:** The set of functors

$$\{ HH_{\mathcal{V}}(A;-) : \mathcal{R}ing_{\mathcal{V}}(A,A) \longrightarrow Ho\mathcal{V} \mid A \in (\mathcal{R}ing_{\mathcal{V}})_0 \}$$

can be equipped with the structure of a $Ho\mathcal{V}$-shadow on $\mathcal{R}ing_{\mathcal{V}}$.

Key: Dennis-Morita-Waldhausen argument

# Morita invariance

**Cor:** If $A$ and $B$ are Morita equivalent, then $HH_\bullet(A) \cong HH_\bullet(B)$.

# Many-objects generalization

$(\mathcal{V}, \otimes, I)$ very nice

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# Many-objects generalization

$(\mathcal{V}, \otimes, I)$ very nice

- Monoid in $\mathcal{V}$ $\rightsquigarrow$ $\mathcal{V}$-enriched cat $\mathcal{A}$
- $(\mathcal{A}, \mathcal{B})$-bimodule $\rightsquigarrow$ $\mathcal{A}^{op} \otimes \mathcal{B} \longrightarrow \mathcal{V}$

$\mathcal{V}$-enriched