Many-objects generalization

$(\mathcal{V}, \otimes, I)$ very nice

- Monoid in $\mathcal{V}$ $\rightsquigarrow$ $\mathcal{V}$-enriched cat $\mathcal{A}$
- $(\mathcal{A}, \mathcal{B})$-bimodule $\rightsquigarrow$ $\mathcal{A}^{op} \otimes \mathcal{B} \longrightarrow \mathcal{V}$

  $\mathcal{V}$-enriched
- $\mathcal{R}ing_{\mathcal{V}}$ $\rightsquigarrow$ $Cat_{\mathcal{V}}$

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- (A, B)-bimodule $\rightsquigarrow$ $\mathcal{A}^{op} \otimes \mathcal{B} \longrightarrow \mathcal{V}$ ← $\mathcal{V}$-enriched
- $\mathcal{R}ing_{\mathcal{V}}$ $\rightsquigarrow$ $\mathcal{C}at_{\mathcal{V}}$
- $HH_{\mathcal{V}}$ $\rightsquigarrow$ $HH_{\mathcal{V}cat}(\mathcal{A}; \mathcal{M}) \in \mathcal{V}$

$$HH_{\mathcal{V}cat}(\mathcal{A}) := HH_{\mathcal{V}cat}(\mathcal{A}; \mathcal{A})$$

Many-objects generalization

$$\Rightarrow \{ HH_{\mathcal{V}Cat}(\mathcal{A};-) : Cat_{\mathcal{V}}(\mathcal{A},\mathcal{A}) \to Ho\mathcal{V} \mid \mathcal{A} \in (Cat_{\mathcal{V}})_0 \}$$

– a $Ho\mathcal{V}$-shadow on $Cat_{\mathcal{V}}$!

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Agreement: $HH_{\mathcal{V}}(A) \cong HH_{\mathcal{V}Cat}(\mathcal{Q}Perf_A)$

$\forall A \in (\mathcal{R}ing)_0$.

Universality [Berman], [H-Rasekh]

$\exists$ theory of Hochschild homology of bicategories st $\forall$ bicategory $\mathcal{B}$, $\exists$ $\mathrm{biHH}(\mathcal{B})$-shadow on $\mathcal{B}$

$$\{\mathcal{B}(a,a) \longrightarrow \mathrm{biHH}(\mathcal{B}) \mid a \in \mathcal{B}_0\}$$

that is universal:

$$\{\underline{C}\text{-shadows on } \mathcal{B}\} \cong \mathrm{Fun}(\mathrm{biHH}(\mathcal{B}), \underline{C}).$$

Applications to equivariant theories

Encoding cyclic actions

$(\mathcal{V}, \otimes, I)$ as before

Let $\mathcal{V}_{aut} = \mathrm{Fun}(B\mathbb{Z}, \mathcal{V})$.

objects : (v, φ)

$v \in \mathcal{V}$

$\varphi \in \mathrm{Aut}_{\mathcal{V}}(v)$

Encoding cyclic actions

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Let $\mathcal{V}_{aut} = \mathrm{Fun}(BC_\infty, \mathcal{V})$.

objects : (v, φ) $v \in \mathcal{V}$, $\varphi \in \mathrm{Aut}_{\mathcal{V}}(v)$

Ex: $\mathcal{V} = \mathrm{Sp}$, X orthogonal C_n-spectrum,

g generator of C_n $\Rightarrow$ $g \in \mathrm{Aut}_{\mathrm{Sp}}(X)$, i.e., $(X, g) \in \mathrm{Sp}_{aut}$

Encoding cyclic actions

$(\mathcal{V}, \otimes, \mathbb{I})$ as before

Let $\mathcal{V}_{aut} = \mathrm{Fun}(B\mathbb{Z}_\infty, \mathcal{V})$.

$\Rightarrow \big(\mathcal{V}_{aut}, \otimes, (\mathbb{I}, \mathrm{Id}_{\mathbb{I}})\big)$ also nice, whence:

Encoding cyclic actions

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Let $\mathcal{V}_{aut} = \mathrm{Fun}(B\mathbb{C}_\infty, \mathcal{V})$.

$\Rightarrow (\mathcal{V}_{aut}, \otimes, (\mathbb{I}, \mathrm{Id}_{\mathbb{I}}))$ also nice, whence:

Prop: $\exists$ $\mathrm{Ho}(\mathcal{V}_{aut})$-shadow

$$\{ HH_{\mathcal{V}_{aut}}((A,\varphi); -) : \mathrm{Ring}_{\mathcal{V}_{aut}}((A,\varphi),(A,\varphi)) \to \mathrm{Ho}\,\mathcal{V}_{aut} \}_{(A,\varphi)}$$

Comparison with twisted THH

For $\mathcal{V} = Sp$, R orthogonal C_n-ring spectrum can certainly compute

$$HH_{Sp_{aut}}(R) \in Sp_{aut}$$

but $HH_{Sp_{aut}}(R) \not\simeq THH_{C_n}(R)$ in general.

The correct generalization

(A,φ) monoid in $\mathcal{N}_{aut}$, (M,γ) an (A,φ)-bimod:

$$\begin{tikzcd}
A\otimes M \arrow[r, "\lambda"] \arrow[d, "\varphi\otimes M"'] \arrow[dr, phantom, "\circlearrowright"] & M \arrow[d, "\gamma"] \arrow[dr, phantom, "\circlearrowright"] & M\otimes A \arrow[l, "\rho"'] \arrow[d, "\gamma\otimes\varphi"] \\
A\otimes M \arrow[r, "\lambda"] & M & M\otimes A \arrow[l, "\rho"']
\end{tikzcd}$$

The correct generalization

(A,φ) monoid in $\mathcal{N}_{aut}$, (M,Γ) an (A,φ)-bimod

$\Rightarrow \quad {}^{\varphi}M = (M, \lambda \circ (\varphi \otimes \mathrm{Id}_M), \rho)$

$\quad\quad M^{\varphi^{-1}} = (M, \lambda, \rho \circ (\varphi^{-1} \otimes \mathrm{Id}_M))$

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$$\gamma : {}^{\varphi}M \xrightarrow{\cong} M^{\varphi^{-1}}$$

as A-bimodules.

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$$HH^{\varphi}(A;(M,\gamma)) := HH_{\mathcal{N}}(A;{}^{\varphi}M)$$

$$HH^{\varphi}(A) := HH^{\varphi}(A;(A,\varphi))$$

The correct generalization

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Ex: $\mathcal{V} = Sp \Rightarrow THH_{C_n}(R) = HH^{g}(A)$

Properties of HH^{φ}

- If (A,φ), (B,ψ) Morita equivalent, then

$$HH^{\varphi}(A) \cong HH^{\psi}(B)$$

Properties of HH^{φ}

- If (A,φ), (B,ψ) Morita equivalent, then

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BUT

- HH^{φ} does NOT underly a shadow, due to not satisfying cyclic invariance axiom.

Consequences for THH_{C_n}

- Morita invariance:

R, S Morita equivalent orthogonal C_n-spectra

$$\Rightarrow \quad THH_{C_n}(R) \simeq THH_{C_n}(S).$$

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- $\{THH_{C_n}(R;-)^{C_n} \mid R\}$ DOES underly a shadow

Furthermore ...

- Similar story for $\underline{HH}_{C_n}$ of Green functors, where taking C_n-fixed points replaced by evaluating at C_n/C_n.

Furthermore ...

- Similar story for $\underline{HH}_{C_n}$ of Green functors, where taking C_n-fixed points replaced by evaluating at C_n/C_n.
- Can see linearization maps $\pi_k THH \to HH_k$ and analogue for Green functors as morphisms of shadows

<u>And finally</u> ... For (R,φ) monoid in $\mathcal{V}_{aut}$,

$\exists$ generalization of the Dennis trace map

$$K(\mathrm{Perf}_{(R,\varphi)}) \longrightarrow HH_{\mathcal{V}}(R;\, {}^{\varphi}R).$$

Thank you!