Stable homotopy groups

How to compute with the Adams spectral sequence (+ related matters)

Not: • a summary of recent progress

• a report on stable homotopy groups in high dimensions

• a description of the latest techniques: motivic homotopy theory and deformations

• a construction of the Adams spectral sequence (+ related matters), nor convergence issues. [See Rognes's lecture notes]

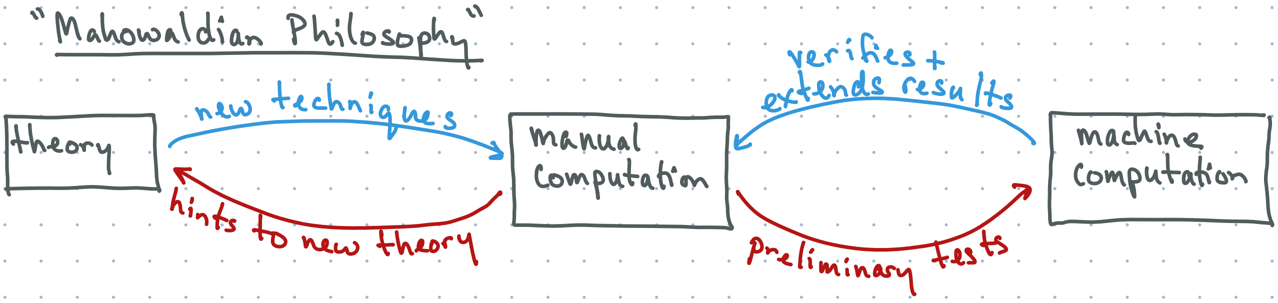
Goals: • practical skills for managing complicated spectral sequences.

• algebraic tools, tricks, methods, + techniques.

• examples

• always 2-primary

"Mahowaldian Philosophy"



Slogan: Computation precedes theory.

Ex: Mahowald observed that $\mathrm{Ext}_{A(2)}(\mathbb{F}_2, \mathbb{F}_2)$ could be an Adams E_2-page, long before tmf was constructed.

Ex: We computed a lot of $\mathbb{C}$-motivic stable homotopy groups before realizing that $\mathbb{C}$-motivic stable homotopy theory is a deformation of classical stable homotopy theory.

Filtered objects + their associated graded objects

A spectral sequence computes the associated graded object of a filtration on the target of the spectral sequence.

This is typically viewed as a shortcoming, a loss of information.

Alternatively, the filtration is rich additional structure that aids in the analysis/computation of the target.

I prefer to work with Adams E_∞-pages, only resolving extensions as needed on the fly.

Note: Rich additional structure is a recurring theme.
The stable homotopy groups, with "all" additional structure, encode the entire homotopy category of (finite? finite-type bounded below?) spectra.

Notation: When indexing a spectral sequence, start with a convenient notation for the target. Then work backwards to whatever that implies about the E_r-pages.

Ex: $\mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_{t-s}$

s = homological degree

t = internal Steenrod algebra degree

I prefer to write

$\mathrm{Ext}_A^{n,s}(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_n$

$n = t-s$ = Adams chart x-coordinate

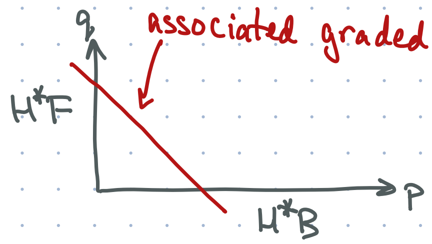
s = homological degree

Adams chart y-coordinate

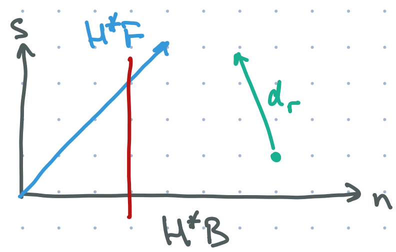
Reindex so that the associated graded objects are aligned in vertical columns.

Ex: Serre spectral sequence

$$H^pB \otimes H^qF \Rightarrow H^{p+q}E$$



$$H^sB \otimes H^{n-s}F \Rightarrow H^nE$$



Maps + Filtrations

Let A and B have decreasing filtrations F_*A and F_*B, with associated graded objects Gr_*A and Gr_*B.

$Gr_pA = F_pA / F_{p+1}A$

Let $x \in A$, $x \neq 0$. Then $x \in F_pA$ but $x \notin F_{p+1}A$ for some p, and $\bar{x} \in Gr_pA$.

Let $f: A \to B[k]$ be a filtration-preserving map, so $f(F_pA) \subseteq F_{p+k}B$

shift the filtration index

This induces $\mathrm{Gr} f : \mathrm{Gr} A \longrightarrow \mathrm{Gr} B[k]$

Generically, if $f(x) \neq 0$ in B, then $\mathrm{Gr} f(\bar{x})$ is non-zero in $\mathrm{Gr} B[k]$.
(the filtration of x equals the filtration of $f(x)$.

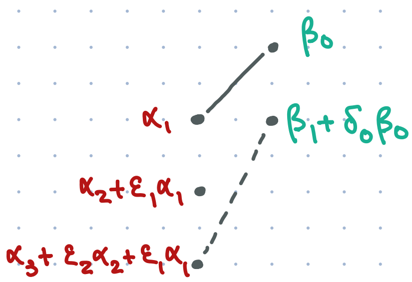
But it is possible that the filtration of $f(x)$ is strictly greater than the filtration of x.

Then $\mathrm{Gr} f(\bar{x}) = 0$, even though $f(x) \neq 0$.

This is called a "hidden value" of f.

<u>Ex</u>: $\eta : \pi_n \longrightarrow \pi_{n+1}$ induces a map $E_\infty \longrightarrow E_\infty[1]$ (Adams E_∞-pages)

A "hidden extension by η" is a hidden value of this map.



$\eta \cdot \alpha_1 = \beta_0$

$\eta(\alpha_2 + \varepsilon_1 \alpha_1) = 0$ for some value of ε_1.

$\eta(\alpha_2 + (1+\varepsilon_1)\alpha_1) = \beta_0$

Use the relation $\eta \alpha_2 = 0$ to define α_2.

$\eta(\alpha_3 + \varepsilon_2 \alpha_2 + \varepsilon_1 \alpha_1) = \beta_1 + \delta_0 \beta_0$ for some $\varepsilon_2, \varepsilon_1, \delta_0$.

Change ε_1 if necessary, so $\eta(\alpha_3 + \varepsilon_2 \alpha_2 + \varepsilon_1 \alpha_1) = \beta_1$.

Then $\eta(\alpha_3 + \varepsilon_2 \alpha_2 + (1+\varepsilon_1)\alpha_1) = \beta_1 + \beta_0$.

<u>Note</u>: Use caution when "defining away" the hidden extensions.
Are the choices for 2, η, and ν extensions compatible?

<u>Ex</u>: $2 : \pi_n \longrightarrow \pi_n$ induces $E_\infty \longrightarrow E_\infty[1]$

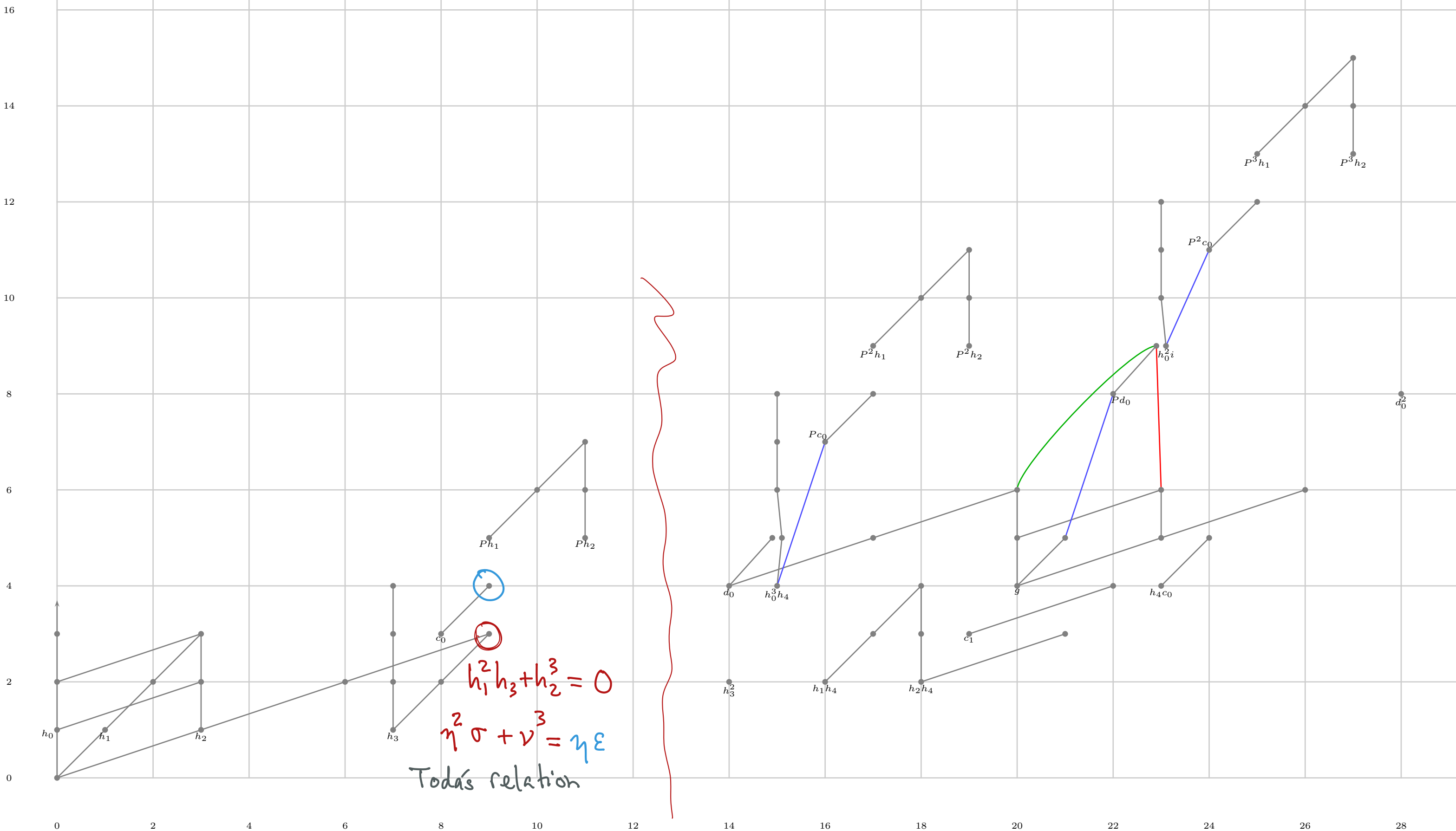
This determines the group structure on π_n.

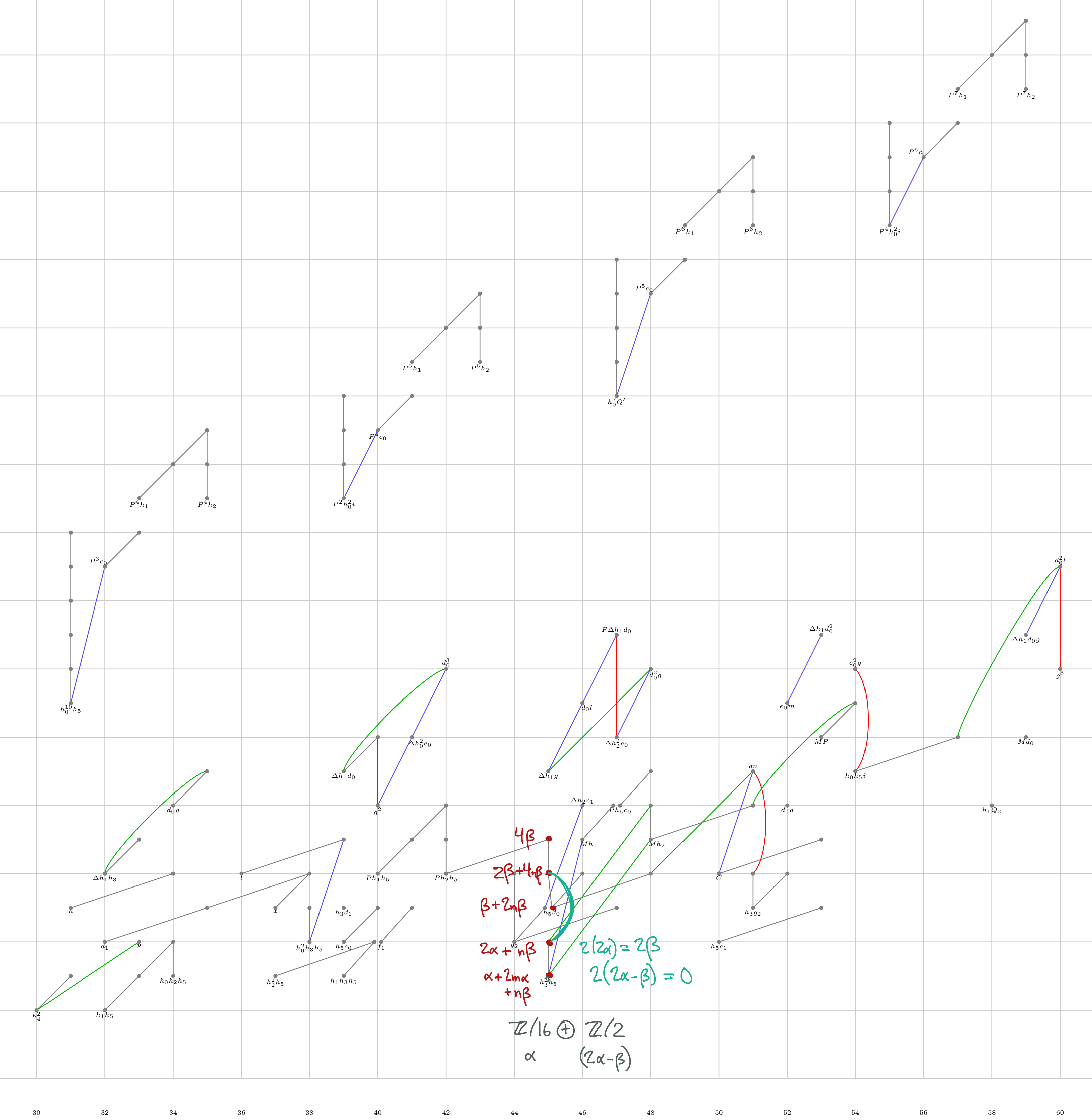
The data ($E_\infty \longrightarrow E_\infty[1]$) plus hidden 2 extensions is preferable.

<u>Ex:</u>

 $= \mathbb{Z}/8 \oplus \mathbb{Z}/4$

 $= \mathbb{Z}/4 \oplus \mathbb{Z}/2$


h_0
h_1
h_2
h_3
c_0
Ph_1
Ph_2
h_3^2
d_0
$h_0^3 h_4$
Pc_0
$h_1 h_4$
$h_2 h_4$
c_1
$P^2 h_1$
$P^2 h_2$
g
Pd_0
$h_4 c_0$
$h_0^2 i$
$P^2 c_0$
$P^3 h_1$
$P^3 h_2$
d_0^2
$h_1^2 h_3 + h_2^3 = 0$
$\eta^2 \sigma + \nu^3 = \eta \varepsilon$
Toda's relation



P^7h_1
P^7h_2
P^6c_0
P^6h_1
P^6h_2
$P^4h_0^2i$
P^5c_0
P^5h_1
P^5h_2
h_0^7Q'
P^4c_0
P^4h_1
P^4h_2
$P^2h_0^2i$
P^3c_0
d_0^2l
$P\Delta h_1d_0$
$\Delta h_1d_0^2$
Δh_1d_0g
d_0^3
d_0^2g
e_0^2g
g^3
$h_0^{10}h_5$
d_0l
e_0m
$\Delta h_0^2e_0$
$\Delta h_2^2c_0$
MP
Md_0
Δh_1d_0
Δh_1g
gn
h_0h_5i
d_0g
g^2
Δh_2c_1
Ph_5c_0
d_1g
h_1Q_2
Mh_1
Mh_2
Δh_1h_3
t
Ph_1h_5
Ph_2h_5
C
n
x
h_3d_1
h_5d_0
h_3g_2
d_1
p
$h_0^2h_3h_5$
h_5c_0
f_1
g_2
h_5c_1
$h_0h_2h_5$
$h_2^2h_5$
$h_1h_3h_5$
$h_3^2h_5$
h_4^2
h_1h_5
4β
$2\beta+4n\beta$
$\beta+2n\beta$
$2\alpha+n\beta$
$\alpha+2m\alpha$
$+n\beta$
$2(2\alpha)=2\beta$
$2(2\alpha-\beta)=0$
$\mathbb{Z}/16 \oplus \mathbb{Z}/2$
α
$(2\alpha-\beta)$
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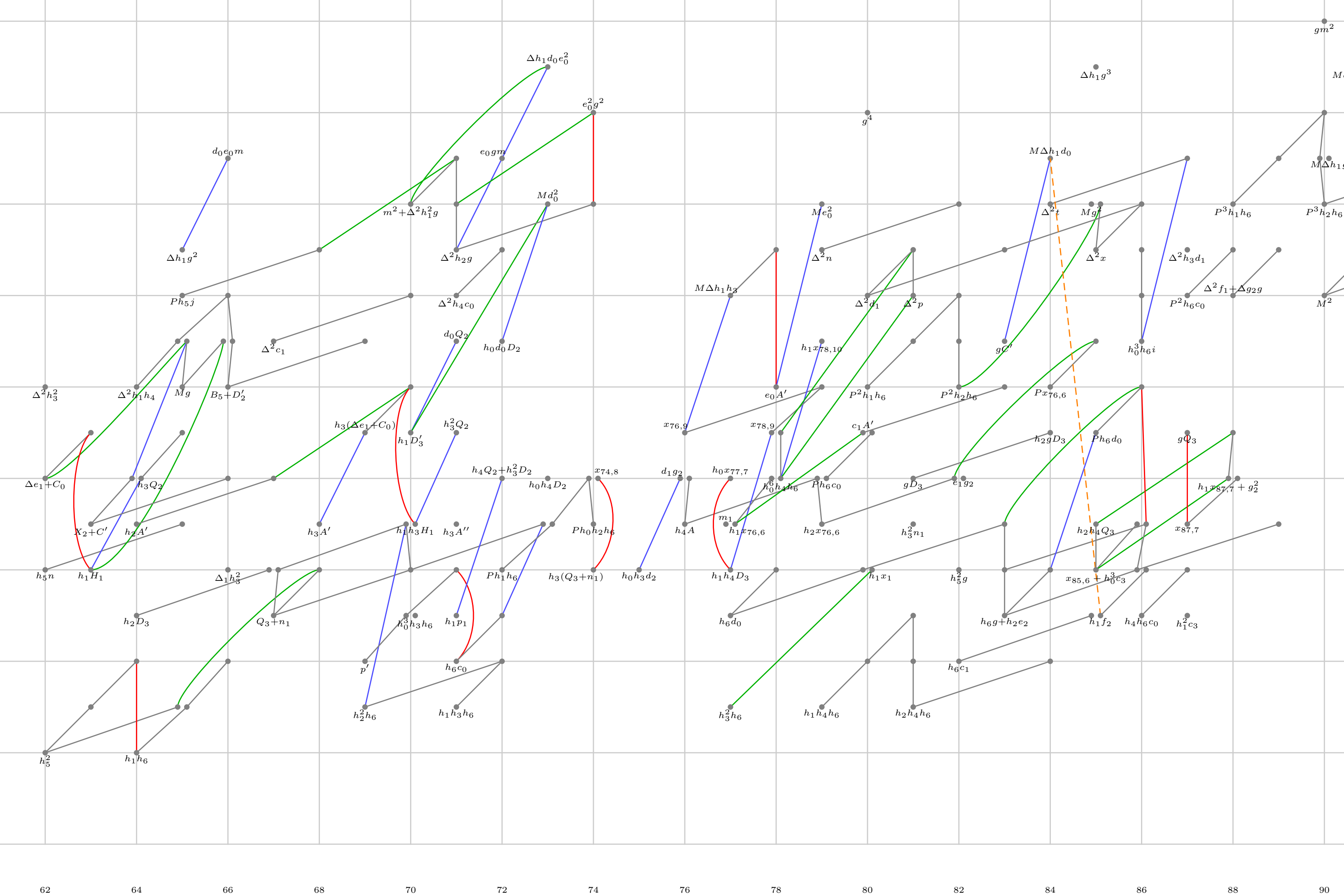
gm^2
$\Delta h_1 d_0 e_0^2$
$\Delta h_1 g^3$
$e_0^2 g^2$
g^4
$d_0 e_0 m$
$e_0 g m$
$M\Delta h_1 d_0$
Md_0^2
$m^2+\Delta^2 h_1^2 g$
Me_0^2
$\Delta^2 t$
Mg^2
$P^3 h_1 h_6$
$P^3 h_2 h_6$
$\Delta h_1 g^2$
$\Delta^2 h_2 g$
$\Delta^2 n$
$\Delta^2 x$
$\Delta^2 h_3 d_1$
$M\Delta h_1 h_3$
$Ph_5 j$
$\Delta^2 h_4 c_0$
$\Delta^2 d_1$
$\Delta^2 p$
$P^2 h_6 c_0$
$\Delta^2 f_1+\Delta g_2 g$
M^2
$d_0 Q_2$
$h_0 d_0 D_2$
$h_1 x_{78,10}$
gC'
$h_0^3 h_6 i$
$\Delta^2 c_1$
$\Delta^2 h_3^2$
$\Delta^2 h_1 h_4$
Mg
B_5+D_2'
$e_0 A'$
$P^2 h_1 h_6$
$P^2 h_2 h_6$
$Px_{76,6}$
$h_3(\Delta e_1+C_0)$
$h_3^2 Q_2$
$h_1 D_3'$
$x_{76,9}$
$x_{78,9}$
$c_1 A'$
$h_2 g D_3$
$Ph_6 d_0$
gQ_3
Δe_1+C_0
$h_3 Q_2$
$h_4 Q_2+h_3^2 D_2$
$h_0 h_4 D_2$
$x_{74,8}$
$d_1 g_2$
$h_0 x_{77,7}$
$h_0^6 h_4 h_6$
$Ph_6 c_0$
gD_3
$e_1 g_2$
$h_1 x_{87,7}+g_2^2$
X_2+C'
$h_2 A'$
$h_3 A'$
$h_1 h_3 H_1$
$h_3 A''$
$Ph_0 h_2 h_6$
$h_4 A$
m_1
$h_1 x_{76,6}$
$h_2 x_{76,6}$
$h_3^2 n_1$
$h_2 h_4 Q_3$
$x_{87,7}$
$h_5 n$
$h_1 H_1$
$\Delta_1 h_3^2$
$Ph_1 h_6$
$h_3(Q_3+n_1)$
$h_0 h_3 d_2$
$h_1 h_4 D_3$
$h_1 x_1$
$h_5^2 g$
$x_{85,6}+h_0^3 c_3$
$h_2 D_3$
Q_3+n_1
$h_0^3 h_3 h_6$
$h_1 p_1$
$h_6 d_0$
$h_6 g+h_2 e_2$
$h_1 f_2$
$h_4 h_6 c_0$
$h_1^2 c_3$
p'
$h_6 c_0$
$h_6 c_1$
$h_2^2 h_6$
$h_1 h_3 h_6$
$h_3^2 h_6$
$h_1 h_4 h_6$
$h_2 h_4 h_6$
h_5^2
$h_1 h_6$
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