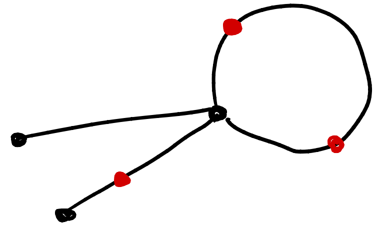
Stable and unstable homology of graph braid groups

(jt w/ B. An + G. Drummond-Cole)

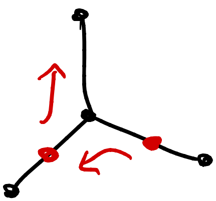
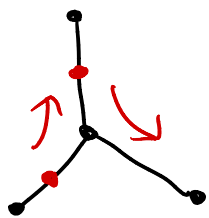
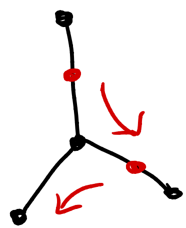
Γ a graph



$\in B_3(\Gamma)$

$$B_k(\Gamma) = \{(x_1,\dots,x_k)\in\Gamma^k \mid x_i\neq x_j \text{ if } i\neq j\}/\Sigma_k$$

Ex $(k=2,\ \Gamma = \lambda)$ $\qquad S^1 \overset{\sim}{\hookrightarrow} B_2(\lambda)$

[ +  + ] $\neq 0 \in \pi_1$

Thm (Abrams) $B_k(\Gamma)$ is aspherical.

Ex (Ghrist) $B_2(K_5) \simeq \#_7 \mathbb{RP}^2$ "graph braid groups"

Def The i^{th} Ramos number of Γ is

$$\Delta^i_\Gamma = \max_{|W|=i} |\pi_0(\Gamma \setminus W)|,$$

where W runs over sets of essential vertices (valence ≥ 3).

Thm (ADK) Fix a field $\mathbb{F}$ and $i > 1$. If Γ is a connected graph with an essential vertex, then

$$\dim H_i(B_k(\Gamma);\mathbb{F}) \sim \left[\sum_{\substack{|W|=i \\ \Delta_0 = \Delta_\Gamma^i}} \frac{1}{(\Delta_\Gamma^i - 1)!} \prod_{w \in W} (d(w)-2) \right] k^{\Delta_\Gamma^i - 1}.$$

In fact, we show the Betti numbers are eventually polynomial in k, so the theorem amounts to calculating the degrees and leading coefficients.

<u>Question</u> Why eventual polynomial growth?

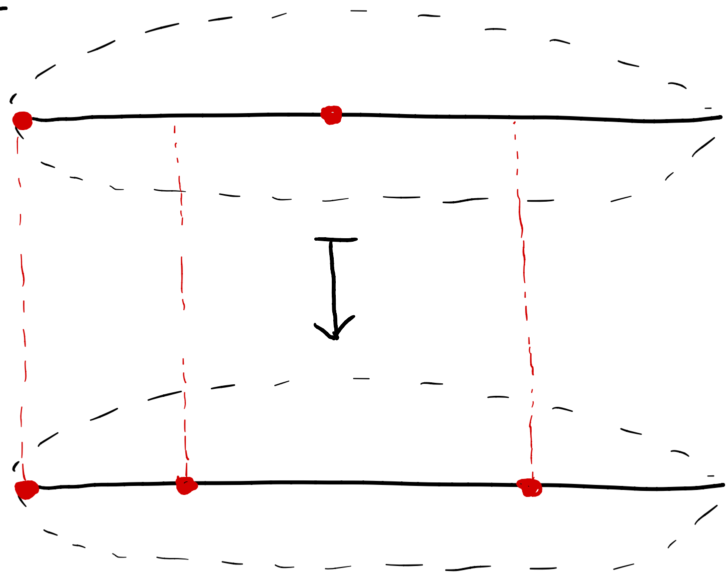
<u>Thm</u> (Hilbert) A finitely generated graded module over $\mathbb{F}[x_1, \dots, x_n]$ exhibits eventual polynomial growth of degree $\leq n-1$.

$e \in E = E(\Gamma)$

$\sigma_e : B_k(\Gamma) \to B_{k+1}(\Gamma)$

$H_*(B(\Gamma)) \circlearrowleft \mathbb{Z}[E]$

$B(\Gamma) := \coprod_{k \geq 0} B_k(\Gamma)$



<u>Thm</u> (ADK) $H_*(B(\Gamma))$ is finitely generated over $\mathbb{Z}[E]$.

<u>Perspective</u> Homological stability

<u>Space</u>	<u>Stable homology</u>	<u>Generation</u>
$B_k(M)$	constant	$\mathbb{Z}[\sigma]$
$\mathrm{Conf}_k(M)$	constant characterwise	FI
$B_k(\Gamma)$	polynomial	$\mathbb{Z}[E]$
$\mathrm{Conf}_k(\Gamma)$	?	?

Question Why degree $\Delta^i_\Gamma - 1$?

To bound the degree from below, we identify a submodule with this growth.

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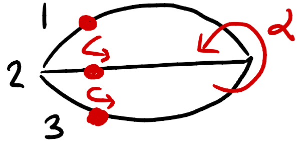
essential vertex $v \in \Gamma$ $\rightsquigarrow$ $\curlywedge \hookrightarrow \Gamma$ $\rightsquigarrow$ "star class" in $H_1(B_2(\Gamma))$

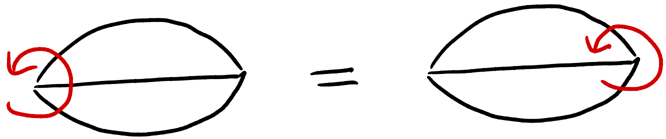
set W of essential vertices $\rightsquigarrow$ $\coprod_W \curlywedge \hookrightarrow \Gamma$ $\rightsquigarrow$ "W-torus" in $H_{|W|}(B_{2|W|}(\Gamma))$

Observation The action of $\mathbb{Z}[E]$ on a W-torus α factors through $\mathbb{Z}[E] \twoheadrightarrow \mathbb{Z}[\pi_0(\Gamma \setminus W)]$.

Ex 

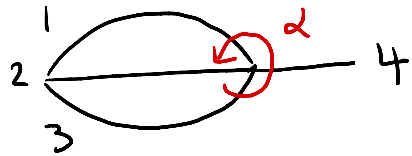
$e_1\alpha = e_2\alpha = e_3\alpha$

Sometimes the action factors further.

Ex 

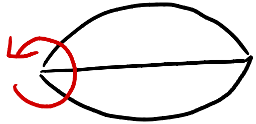
"θ-relation"

$\Longrightarrow$ $e_1\alpha = e_2\alpha = e_3\alpha = e_4\alpha$

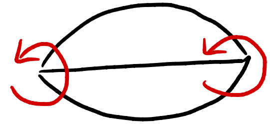


We say that a W-torus is <u>rigid</u> if each star factor intersects at least two components of $\Gamma \setminus W$.

Ex



not rigid



rigid

Prop (1) The W-torus α is rigid iff $\mathbb{Z}[\pi_0(\Gamma \setminus W)] \twoheadrightarrow \mathbb{Z}[E]\cdot\alpha$ is an isomorphism.

(2) If W maximizes $|\pi_0(\Gamma \setminus W)|$, then a rigid W-torus exists.

This proposition implies that the degree of growth is at least $\Delta_\Gamma^i - 1$:

$$\operatorname{rk} H_i(B_k(\Gamma)) \geq \operatorname{rk} \mathbb{Z}[\pi_0(\Gamma \setminus W)]_{k-2i}$$

$$= \binom{k-2i+|\pi_0(\Gamma \setminus W)|-1}{|\pi_0(\Gamma \setminus W)|-1}$$

$$= \binom{k-2i+\Delta_\Gamma^i-1}{\Delta_\Gamma^i-1}$$

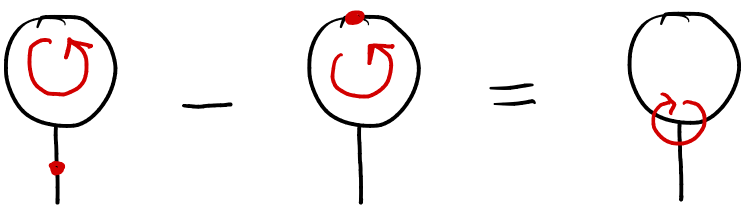
$$= \frac{(k-2i+\Delta_\Gamma^i-1)(k-2i+\Delta_\Gamma^i-2)\cdots(k-2i)}{(\Delta_\Gamma^i-1)!}$$

$$\sim \frac{1}{(\Delta_\Gamma^i-1)!}\, k^{\Delta_\Gamma^i-1}$$

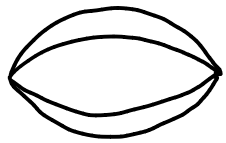
<u>Question</u> What about the upper bound? Why does nothing grow faster than a rigid torus?

If $H_*(B(\Gamma))$ were generated by tori, the Θ-relation would be an answer.

<u>Ex</u> A loop in Γ gives a class in $H_1(B_1(\Gamma))$ not in the span of all tori, but



“Q-relation”

Ex  Θ_4 $\quad B_3(\Theta_4) \simeq \#_3 T^2$

The fundamental class λ is not in the span of all tori, but one can show that $(e_i - e_j)\lambda$ is.

So loop classes and Θ-classes grow slowly modulo tori. The theorem implies that relations of this form hold for all classes.

<u>Perspective</u> Generators and relations

<u>Problem</u> Give a list of atomic graphs generating $H_i(B(\Gamma))$ for some class of graphs Γ.

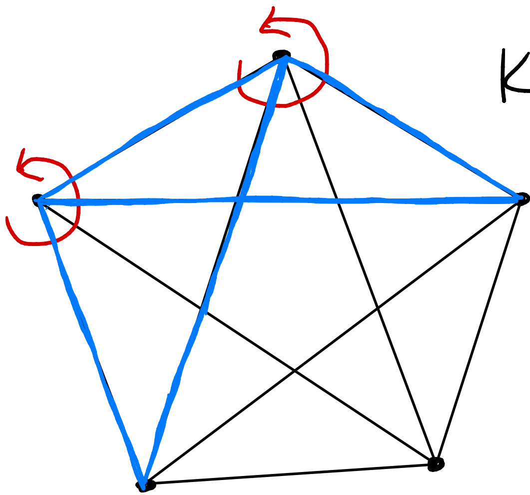
<u>Thm</u> (Ko-Park) For $i=1$ and all graphs, $\{\bigcirc, \curlywedge\}$ is a generating set.

<u>Thm</u> (AK) For $i=2$ and planar graphs, Θ_4 is the only new generator. For non-planar graphs, there are more.

Asymptotically, the only generator is $\curlywedge$.

Perspective Torsion

Thm (Ko–Park) There is (2-)torsion in $H_1(B(\Gamma))$ iff Γ is non-planar.



K_5

Θ-relation $\Longrightarrow \alpha = (-1)^5 \alpha$

No other torsion is known.

<u>Conjecture</u> (?) $H_*(B(\Gamma))$ has no odd torsion and, if Γ is planar, also no even torsion.

Asymptotically, the conjecture holds:

$$\dim H_i(B(\Gamma); \mathbb{F}_p) \sim \dim H_i(B(\Gamma); \mathbb{Q}).$$

Any torsion must arise from exotic classes with slow growth.