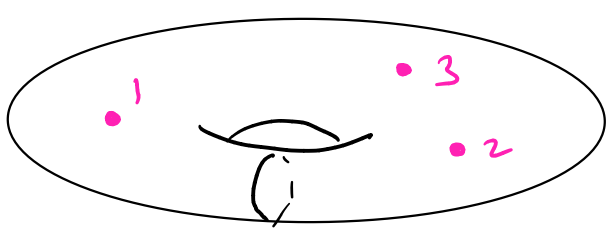
How to build a surface of genus 6

X a space

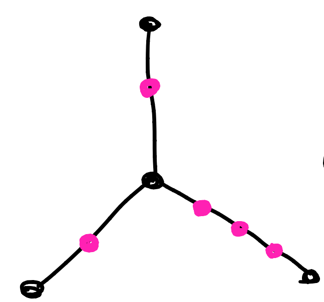
Def The k^{th} ordered <u>configuration space</u> of X is

$$\mathrm{Conf}_k(X) = \{(x_1, \dots, x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\}$$

(resp. unordered, $B_k(X) = \mathrm{Conf}_k(X)/\Sigma_k$).

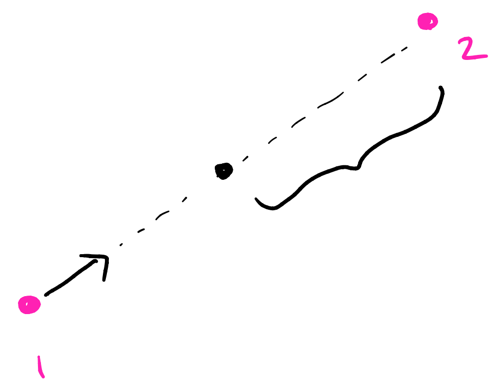


$\in \mathrm{Conf}_3(T^2)$



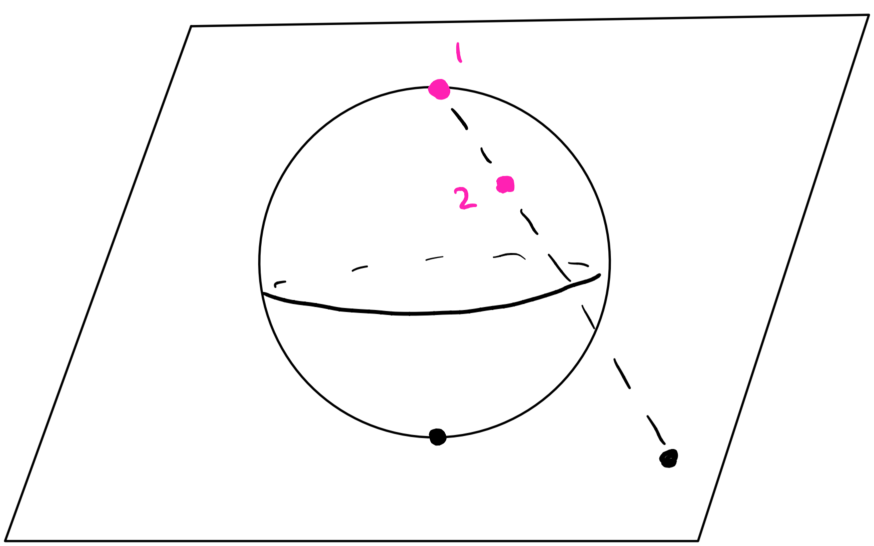
$\in B_5(S_3)$

Ex $X = \mathbb{R}^n$, $k=2$



$$\begin{tikzcd} \mathrm{Conf}_2(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}_{>0} \times S^{n-1} \arrow[r, "\sim"] \arrow[d] & S^{n-1} \arrow[d] \\ B_2(\mathbb{R}^n) \arrow[r, "\sim"] & \mathbb{RP}^{n-1} \end{tikzcd}$$

Ex $X = S^n$, $k=2$

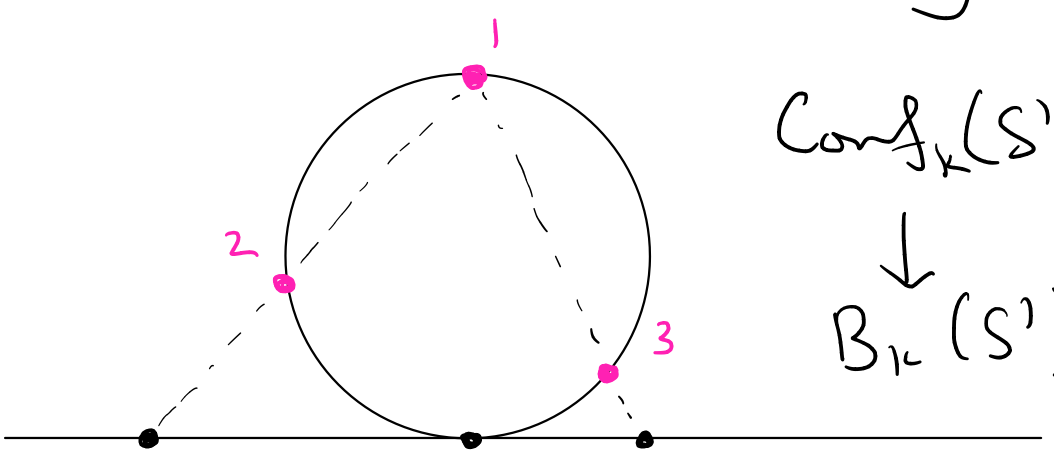


$$\begin{tikzcd} \mathrm{Conf}_2(S^n) \cong TS^n \arrow[r, "\sim"] \arrow[d] & S^n \arrow[d] \\ B_2(S^n) \arrow[r, "\sim"] & \mathbb{RP}^n \end{tikzcd}$$

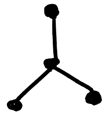
Ex $X = \mathbb{R}$, k arbitrary

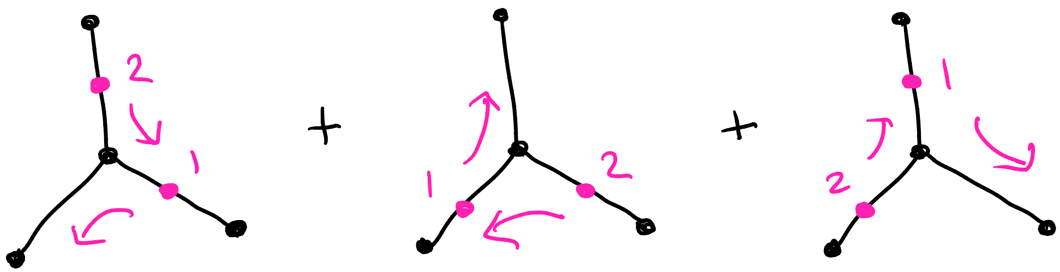
$$
\begin{tikzcd}
\mathrm{Conf}_k(\mathbb{R}) \cong \Sigma_k \times \mathring{\Delta}^k \arrow[r, "\sim"] \arrow[d] & \Sigma_k \arrow[d] \\
B_k(\mathbb{R}) \arrow[r, "\sim"] & pt
\end{tikzcd}
$$

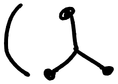
Ex $X = S^1$, k arbitrary



$$
\begin{tikzcd}
\mathrm{Conf}_k(S^1) \cong S^1 \times \mathrm{Conf}_{k-1}(\mathbb{R}^1) \arrow[r, "\sim"] \arrow[d] & S^1 \times \Sigma_{k-1} \arrow[d] \\
B_k(S^1) \arrow[r, "\sim"] & S^1
\end{tikzcd}
$$

Ex $X =$  , $k=2$



$S^1 \longrightarrow \mathrm{Conf}_2($$)$

Does it detect a hole?

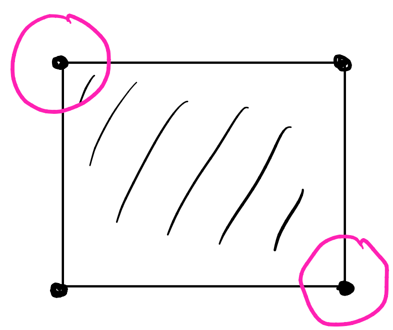
Idea (Abrams) Γ a graph

$$\Gamma^k \supseteq \mathrm{Conf}_k(\Gamma) \supseteq \mathrm{Conf}_k^{\square}(\Gamma) = \bigcup_{\substack{\overline{c_i} \cap \overline{c_j} = \phi \\ \text{if } i \neq j}} c_1 \times \cdots \times c_k$$

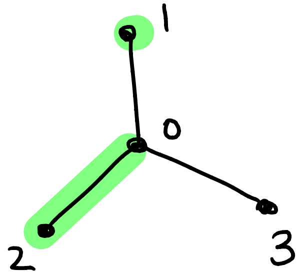
edge or vertex

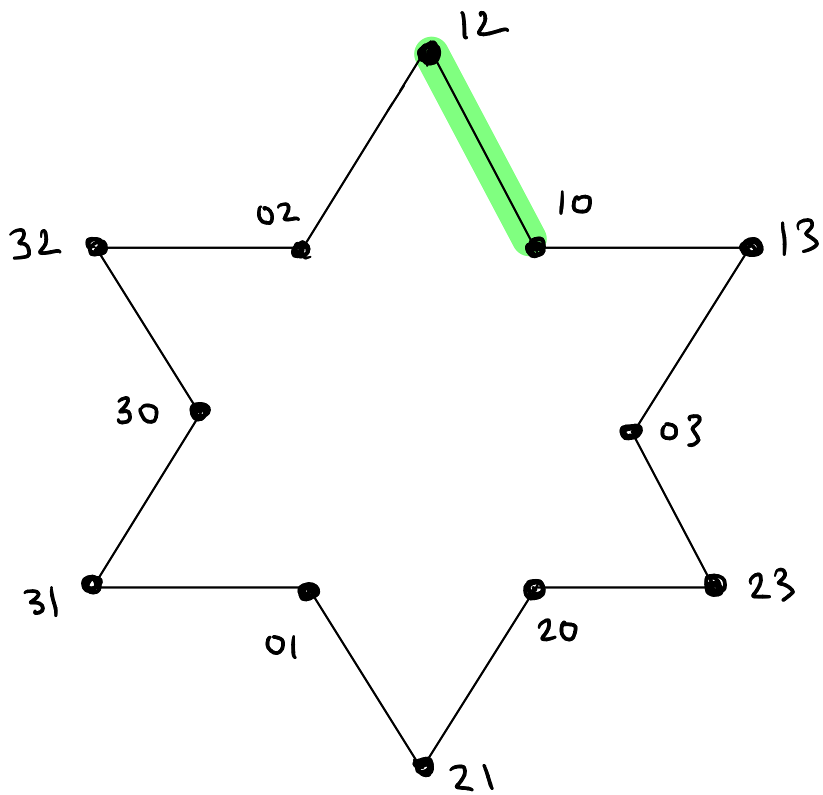
Good approximation if Γ is "sufficiently subdivided"

Ex $\Gamma =$  , $k=2$



Ex $\Gamma = \lambda$, $k=2$



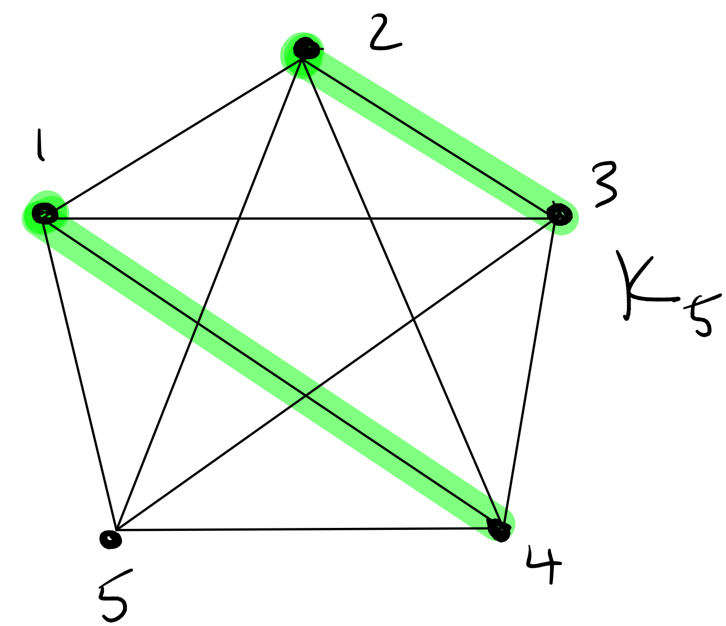


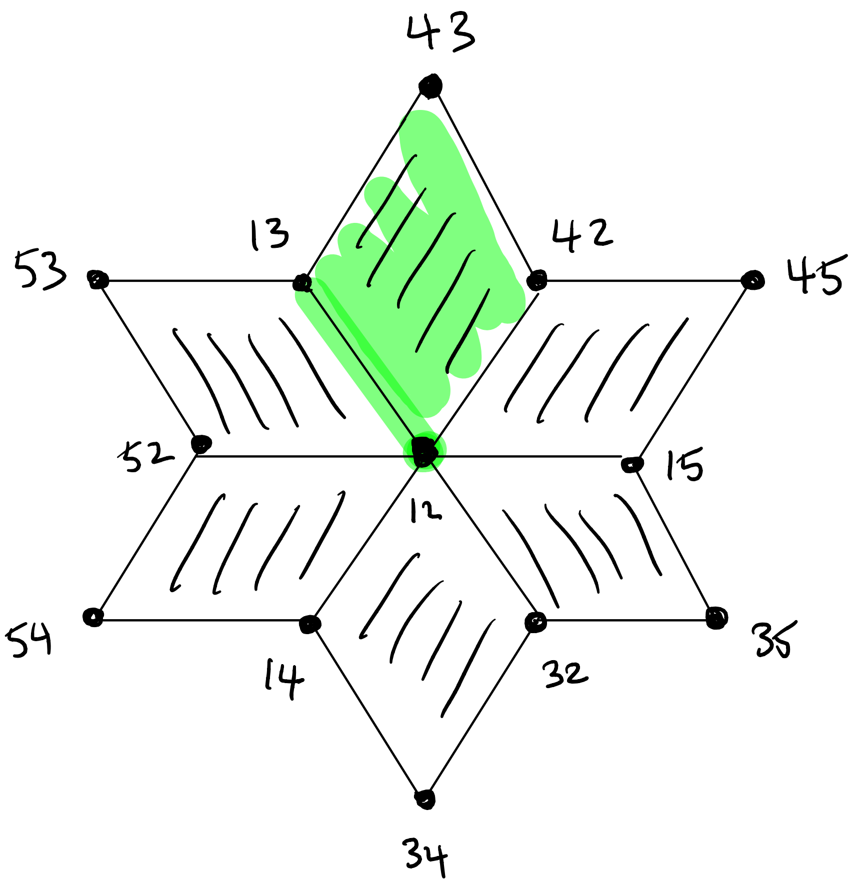
$\cong S^1$

So $\mathrm{Conf}_2(\lambda) \simeq S^1$.

Ex $\Gamma = K_5$, $k=2$

Local picture in $\mathrm{Conf}_2^{\square}(K_5)$ near the vertex 12:



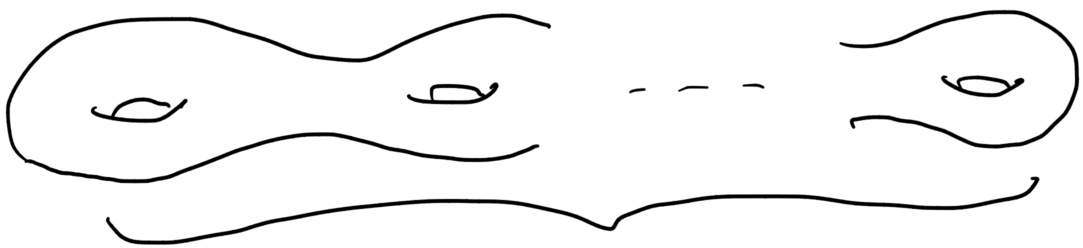


Cells of higher dimension would need ≥ 6 vertices

$\Longrightarrow \mathrm{Conf}_2^{\square}(K_5)$ is a surface!*

* Compact and orientable

Surfaces are classified by genus g, related to Euler characteristic by $\chi = 2 - 2g$.



g

$$\left.\begin{aligned} \#\text{ vertices} &= \binom{5}{2} \cdot 2 = 20 \\ \#\text{ edges} &= \binom{5}{3}\binom{3}{2} \cdot 2 = 60 \\ \#\text{ faces} &= \binom{5}{2}\binom{3}{2} = 30 \end{aligned}\right\} \quad \chi = -10$$

$$\boxed{g = 6}$$

7 ways to love configuration spaces

- Robotics, motion-planning problems, topological complexity (Ghrist, Farber, Lütgehetmann, Recio-Mitter, Bianchi ...)
- Operads, iterated loop spaces, embedding calculus, factorization homology (May, Cohen, de Brito-Weiss, Ayala-Francis, Lurie ...)
- Physics, finite type invariants (Kontsevich, Sinha, Volic, Koytcheff ...)
- Surface and graph braid groups (Artin, Arnold, Farley-Sabalka ...)
- Invariants/invariance (Longoni-Salvatore, Aouina-Klein, Petersen, Ko-Park, Sabalka ...)

- Asymptotics, stability phenomena (McDuff, Church, Ellenberg, Farb, Ramos ...)
- And more ...? (Hahn-Wilson, Chan-Galatius-Payne)