HOMOLOGICAL STABILITY MINICOURSE
LECTURE 1: THE HOMOLOGY OF SYMMETRIC GROUPS

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ABSTRACT. In the first lecture of this minicourse, we introduce the phenomenon of homological stability through the example of the symmetric groups Σ_n. We also explain what one use can these homological stability results for.

1. HOMOLOGY GROUPS OF SYMMETRIC GROUPS

The *symmetric group* Σ_n is the group of bijections of the finite set $\underline{n} = \{1, \ldots, n\}$, under composition. The classifying space BG of a discrete group G, such as Σ_n, is the connected space determined uniquely up to weak homotopy equivalence by the property

$$\pi_*(BG) = \begin{cases} G & \text{if } * = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It can be constructed by extracting from G the groupoid $* /\!\!/ G$ given by:

- a single object $*$,
- morphisms given by $* \xrightarrow{g} *$ for $g \in G$, and
- composition given by multiplication.

We then take its nerve to obtain a simplicial set, and take the geometric realisation to get a topological space $|N(* /\!\!/ G)|$; this is a model for BG. Exercise 3.1 proves it indeed has the desired property.

Question 1.1. What are the homology groups $H_*(B\Sigma_n; \mathbb{Z})$?

Remark 1.2. This is the same as computing the group homology of Σ_n with coefficients in $\mathbb{Z}$ in the sense of [Bro94], see Exercise 3.2.

Let us compute these groups and the homology of their classifying spaces for the first few values of n.

Example 1.3. For $n = 0, 1$, the group Σ_n is trivial so its classifying space is weakly contractible and hence has trivial homology.

Example 1.4. For $n = 2$, Σ_2 is isomorphic to the cyclic abelian group $\mathbb{Z}/2$. Then $B\mathbb{Z}/2$, as constructed above, is homotopy equivalent to $\mathbb{R}P^\infty$. We conclude that

$$H_*(B\mathbb{Z}/2; \mathbb{Z}) = H_*(\mathbb{R}P^\infty; \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0 \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ is odd,} \\ 0 & \text{if } * > 0 \text{ is even.} \end{cases}$$

For an alternative argument, see Exercise 3.3.

Example 1.5. For $n = 3$, the group Σ_3 is the dihedral group D_3 with 6 elements (i.e. the symmetries of a triangle). A more complicated computation given in Exercise 3.5 yields the homology of D_3:

$$H_*(BD_3;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } * = 0, \\ \mathbb{Z}/2 & \text{if } * > 0 \text{ and } * \equiv 1 \pmod 4, \\ \mathbb{Z}/6 & \text{if } * > 0 \text{ and } * \equiv 3 \pmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

As the previous example indicates, direct computation of homology becomes increasingly difficult. However, it is certainly possible by computer.[1] Let's look at Fig. 1 to see whether we can discern some patterns:

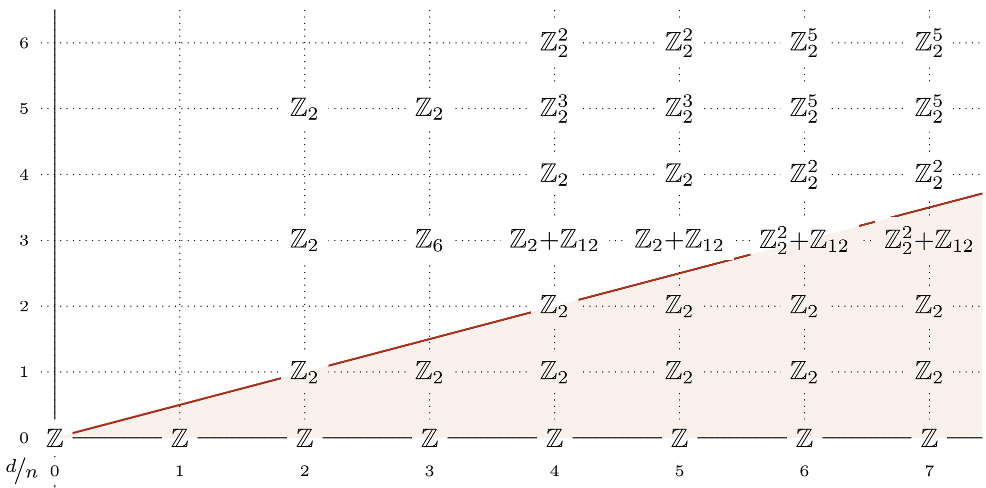

FIGURE 1. The homology groups $H_d(B\Sigma_n;\mathbb{Z})$. To keep this table readable, some compromises had to be made: we wrote $\mathbb{Z}_d^r$ for $(\mathbb{Z}/d)^{\oplus r}$, $+$ for $\oplus$, and combined some summands. The stable range from Theorem 1.8 is shaded.

Here some things one might conjecture after looking at these computations:

(1) Each reduced homology group $\widetilde{H}_d(B\Sigma_n;\mathbb{Z})$ is finite and has small exponent.
(2) The homology in fixed degree $* = d$ becomes independent of n as $n \to \infty$.
(3) Before becoming independent of n, the homology only increases in size.
(4) The p-power torsion only changes when $p|n$.

If we want to attempt to prove (2)–(4), we need a better way to compare the homology groups for different n than just as abstract abelian groups. This is done by observing that the inclusion $\underline{n} \hookrightarrow \underline{n+1}$ of finite sets gives a homomorphism

$$\sigma \colon \Sigma_n \longrightarrow \Sigma_{n+1},$$

by extending a permutation of $\underline{n}$ by the identity on $n+1 \in \underline{n+1}$ to a permutation of $\underline{n+1}$. Our construction of BG is natural in groups and homomorphisms, so this

[1] We in fact know all homology groups of all symmetric groups, in the sense that there is a mechanical procedure for determining them. This can be done combining the work of Nakoaka with that of May [CLM76].

homomorphism induces a map

$$\sigma \colon B\Sigma_n \longrightarrow B\Sigma_{n+1},$$

which in turn induces a map $\sigma_* \colon H_*(B\Sigma_n;\mathbb{Z}) \to H_*(B\Sigma_{n+1};\mathbb{Z})$ on homology. We can then give sharper formulations of (2)–(4) in terms of these *stabilisation maps*:

(2') The maps σ_* are isomorphisms in a range increasing with n.
(3') The maps σ_* are injective.
(4') The maps σ_* are isomorphisms on p-power torsion unless $p|n+1$.

Property (1) holds for all finite groups, and the result which proves it also implies (4'):

Proposition 1.6. *For a finite group G, $\widetilde{H}_*(BG;\mathbb{Z}[1/|G|]) = 0$. More generally, for $H \subset G$ the map $\iota_* \colon H_*(BH;\mathbb{Z}[1/[G:H]]) \to H_*(BG;\mathbb{Z}[1/[G:H]])$ admits a right inverse τ (i.e. $\iota_* \circ \tau = \mathrm{id}$).*

Proof. The first statement follows from the second by taking $H = \{e\}$. The second is proven in Exercise 3.4 using transfer maps. ☐

To deduce (4') from Proposition 1.6, note that $[\Sigma_{n+1} : \Sigma_n] = n+1$ so by the long exact sequence on homology groups so that $H_*(B\Sigma_n;\mathbb{Z}) \to H_*(B\Sigma_{n+1};\mathbb{Z})$ is surjective after inverting $n+1$. Now set $n+1$ equal to p and invoke (3').

It is phenomenon indicated by (2') that is the subject of this minicourse:

Definition 1.7. A sequence $X_0 \xrightarrow{\sigma} X_1 \xrightarrow{\sigma} X_2 \xrightarrow{\sigma} \cdots$ exhibits *homological stability* if the maps $\sigma_* \colon H_*(X_n;\mathbb{Z}) \to H_*(X_{n+1};\mathbb{Z})$ are isomorphisms in a range of degrees $*$ increasing with n.

In the next two lectures we will prove the following result, due to Nakaoka [Nak60] (though he proved much more):

Theorem 1.8. *The sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely, the induced map*

$$\sigma_* \colon H_*(B\Sigma_n;\mathbb{Z}) \longrightarrow H_*(B\Sigma_{n+1};\mathbb{Z})$$

is surjective if $* \leq \frac{n}{2}$ and an isomorphism if $* \leq \frac{n-1}{2}$.

Remark 1.9. Of course, if we know property (3') holds then the range in the previous theorem in which σ_* is an isomorphism improves to $* \leq \frac{n}{2}$. However, property (3') is rather special—related to the existence of transfer maps—and you should not expect it to hold for general sequences of classifying spaces of groups. We will not comment on it again, but see Exercise 3.6.

Remark 1.10. The ranges in the previous remark are optimal among those of the form $* \leq an + b$ with $a, b \in \mathbb{Q}$.

2. Using homological stability for symmetric groups

Homological stability is a structural property of a sequence of groups, or more generally topological spaces, but it is also useful *tool*. In fact, many homological stability theorems are proven in service of obtaining other mathematical results. To illustrate this, I now want to explain some straightforward applications of Theorem 1.8. These concern the transfer of information from low n to high n and vice-versa. They can be obtained by other methods as well, but their generalisations to other sequences of groups often can not.

2.1. **Alternating groups.** Recall that for path-connected X, the Hurewicz map $\pi_1(X) \to H_1(X;\mathbb{Z})$ coincides with abelianisation (we are suppressing the basepoint). In particular, the map $G \to H_1(BG;\mathbb{Z})$ induces an isomorphism $G^{\mathrm{ab}} \to H_1(BG;\mathbb{Z})$ naturally in G. Thus we can understand the abelianisation of Σ_n by computing its first homology group.

The sign homomorphism $\mathrm{sign}\colon \Sigma_n \to \mathbb{Z}/2$ yields a map

$$\mathrm{sign}\colon B\Sigma_n \longrightarrow B\mathbb{Z}/2,$$

which induces a map on homology. This is compatible with stabilisation, in the sense that $\mathrm{sign} \circ \sigma = \mathrm{sign}$, so we get a commutative squares

$$\begin{tikzcd} H_1(B\Sigma_{n-1};\mathbb{Z}) \arrow[r, "\sigma_*"] \arrow[d, "\mathrm{sign}"] & H_1(B\Sigma_n;\mathbb{Z}) \arrow[d, "\mathrm{sign}"] \\ \mathbb{Z}/2 \arrow[r, equal] & \mathbb{Z}/2. \end{tikzcd}$$

The map $H_1(B\Sigma_2;\mathbb{Z}) \to \mathbb{Z}/2$ is an isomorphism because $\mathrm{sign}\colon \Sigma_2 \to \mathbb{Z}/2$ is. By Theorem 1.8, in the commutative diagram

$$\begin{tikzcd} H_1(B\Sigma_2;\mathbb{Z}) \arrow[r, two heads] \arrow[d, "\cong"'] & H_1(B\Sigma_3;\mathbb{Z}) \arrow[r, "\cong"] \arrow[d] & H_1(B\Sigma_4;\mathbb{Z}) \arrow[r, "\cong"] \arrow[d] & \cdots \\ \mathbb{Z}/2 \arrow[r, equal] & \mathbb{Z}/2 \arrow[r, equal] & \mathbb{Z}/2 \arrow[r, equal] & \cdots \end{tikzcd}$$

the right-most top horizontal map is surjective and the other top horizontal maps are isomorphisms. A single diagram chase then deduces from the fact that the left-most vertical map is an isomorphism that all other vertical maps are.

Thus we have used homological stability to prove that

$$\mathrm{sign}\colon \Sigma_n \longrightarrow \mathbb{Z}/2$$

is the abelianisation for $n \geq 2$, or equivalently that the kernel of the sign homomorphism is exactly the subgroup $[\Sigma_n, \Sigma_n]$ generated by commutators. Recalling that this kernel is exactly the alternating group A_n, we conclude that:

Theorem 2.1. $[\Sigma_n, \Sigma_n] = A_n$.

Remark 2.2. This is a fact you likely knew already, and elementary group-theoretic arguments exist. We could have used this fact instead to give an elementary proof of Theorem 1.8 in degree $* = 1$.

2.2. **Group completion.** Homological stability implies that for in fixed degree $*$, for n sufficienty large the canonical map

$$H_*(B\Sigma_n;\mathbb{Z}) \longrightarrow \operatorname*{colim}_{n\to\infty} H_*(B\Sigma_n;\mathbb{Z})$$

is an isomorphism; the right hand side is known as the *stable homology*. This has two somewhat tautological consequences:

(1) We can compute the right side from the left side.
(2) We can compute the left side from the right side.

This is particularly interesting because the stable homology on the right side has a more familiar description.

When we constructed the stabilisation map, we used that inclusion $\underline{n} \to \underline{n+1}$ yields a homomorphism $\Sigma_n \to \Sigma_{n+1}$. More generally, disjoint union induces a homomorphism $\Sigma_n \times \Sigma_m \to \Sigma_{n+m}$, which yields "multiplication" maps

$$B\Sigma_n \times B\Sigma_m \longrightarrow B\Sigma_{n+m},$$

making the space $\bigsqcup_{n\geq 0} B\Sigma_n$ into a unital topological monoid (these are associative but not commutative, and it is probably better to say E_1-space since that is a homotopy-invariant notion).

Example 2.3. The topological monoid structure makes $\pi_0 \coloneqq \pi_0(\bigsqcup_{n\geq 0} B\Sigma_n)$ into a unital monoid, and this can be identified with $\mathbb{N}$ with its usual addition. That addition is commutative reflects the fact that $\bigsqcup_{n\geq 0} B\Sigma_n$ is homotopy-commutative.

The stabilisation maps assemble to a map $\sqcup_{n\geq 0} B\Sigma_n \to \sqcup_{n\geq 0} B\Sigma_n$, mapping the nth component to the $(n+1)$st, which can equivalently be described a multiplication by an element of $B\Sigma_1$. This makes $H_*(\sqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})$ into $\mathbb{Z}[\pi_0]$-module. On $\mathrm{colim}_\sigma\, H_*(\sqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})$ multiplication by σ is invertible (this map induces a pro-isomorphism on diagrams), so there is a map

$$H_*(\sqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})[\pi_0^{-1}] \longrightarrow \mathrm{colim}_\sigma\, H_*(\sqcup_{n\geq 0} B\Sigma_n;\mathbb{Z}), \tag{1}$$

and it is an algebraic fact about computing localisations that this is an isomorphism.

The right hand side of (1) is nothing but a $\mathbb{Z}$-indexed direct sum of copies of $\mathrm{colim}_n\, H_*(B\Sigma_n;\mathbb{Z})$. Thus the stable homology is equal to one of these summands of $H_*(\sqcup_{n\geq 0} B\Sigma_n;\mathbb{Z})[\pi_0^{-1}]$. The McDuff–Segal group completion theorem tells us when we can equivalently make the path-components into a group *before* passing to homology [MS76, Proposition 1] (see also [ERW19, Section 6]):

Theorem 2.4 (McDuff–Segal)**.** *If M is a homotopy-commutative unital associative topological monoid, then $H_*(M;\mathbb{Z})[\pi_0^{-1}] \cong H_*(\Omega BM;\mathbb{Z})$.*

Here Ω denote the based loop space construction, and BM is the bar construction of M obtained as $|N(* /\!\!/ M)|$, where $* /\!\!/ M$ is the category enriched in topological spaces given by:

- a single object $*$,
- space of morphisms given by M,
- composition given by multiplication.

(Strictly speaking, for this to have the correct homotopy type, the inclusion $\{\mathrm{id}\} \hookrightarrow M$ needs to be a cofibration, a condition satisfied in particular when $M = \bigsqcup_{n\geq 0} BG_n$ with each G_n discrete.) The bar construction BM has a canonical basepoint provided by the object $*$, and that is where our loops are based. There are many techniques, known as *infinite loop space machines* [MT78], to compute the homotopy type of BM. In particular, these apply to $M = \bigsqcup_{n\geq 0} B\Sigma_n$ [BP72, Seg74]:

Theorem 2.5 (Barratt–Priddy–Quillen–Segal)**.** *There is a homotopy equivalence $\Omega B(\bigsqcup_{n\geq 0} B\Sigma_n) \simeq \Omega^\infty \mathbb{S}$.*

Here $\mathbb{S}$ is the *sphere spectrum*, described by saying its nth stage is $\mathbb{S}_n = S^n$ and the map $\Sigma\mathbb{S}_n \to \mathbb{S}_{n+1}$ is the usual identification of ΣS^n with S^{n+1}. In particular $\pi_i(\mathbb{S})$ is the ith *stable homotopy group of spheres* given explicitly as $\mathrm{colim}_{k\to\infty}\pi_{i+k}(S^k)$.

Corollary 2.6. *Let $\Omega_0^\infty\mathbb{S} \subset \Omega^\infty\mathbb{S}$ denote the basepoint component, then*

$$\operatorname*{colim}_{n\to\infty} H_*(B\Sigma_n;\mathbb{Z}) \cong H_*(\Omega_0^\infty\mathbb{S};\mathbb{Z}).$$

Remark 2.7. A fancier way of stating the Barratt–Priddy–Quillen–Segal theorem is that the sphere spectrum $\mathbb{S}$ is the algebraic K-theory spectrum of the category of finite sets. An even fancier way of stating this would involve the "field with one element".

2.3. **Serre's finiteness theorem and variations.** Let us now use Corollary 2.6. By (1) the groups $H_*(B\Sigma_n;\mathbb{Z})$ are finite for $* > 0$. By Theorem 1.8 the same is true for the stable homology as long as restrict to degrees $* \leq \frac{n}{2}$. Since n is arbitrary, the stable homology is finite in all positive degrees. This has the following consequence:

Theorem 2.8 (Serre). *$\pi_*(\mathbb{S})$ is finite for all $* > 0$.*

Proof. By construction, the path-component $\Omega^\infty_0\mathbb{S} \subset \Omega^\infty\mathbb{S}$ corresponding to $0 \in \pi_0(\mathbb{S}) \cong \mathbb{Z}$ (all are homotopy-equivalent) is an infinite loop space and hence a so-called *simple space* (i.e. π_1 is abelian and acts trivially on the higher homotopy groups). Thus we can apply the Hurewicz theorem modulo the Serre class $\mathcal{C}$ of finite abelian groups. This in particular says that $\pi_*(\Omega^\infty_0\mathbb{S})$ is finite for $* \leq d$ if and only if $\widetilde{H}_*(\Omega^\infty_0\mathbb{S};\mathbb{Z})$ is finite for $* \leq d$. Now use that Corollary 2.6 identifies $H_*(\Omega^\infty_0\mathbb{S};\mathbb{Z})$ with the stable homology of symmetric groups. $\square$

We can say something similar about torsion: by (4') the groups $H_*(B\Sigma_n;\mathbb{Z})$ contain no p-torsion for when $n < p$. By Theorem 1.8 we conclude that same is true for the stable homology as long as we restrict to degrees $* \leq \frac{p-1}{2}$. This has the following consequence, working modulo the Serre class $\mathcal{C}$ of finite abelian groups which only have ℓ-torsion for primes $\ell \neq p$:

Proposition 2.9. *$\pi_*(\mathbb{S})$ has no p-torsion for $* \leq \frac{p-1}{2}$.*

In fact, the following better result is known (and this range is optimal):

Theorem 2.10 (Serre). *$\pi_*(\mathbb{S})$ has no p-torsion for $* < 2p-3$.*

Arguing in the other direction, we obtain that the stable homology of symmetric groups has no p-torsion for $* < 2p-3$ and conclude using Corollary 2.6 and property (3') that:

Proposition 2.11. *$H_*(B\Sigma_n;\mathbb{Z})$ has no p-torsion for $* < 2p-3$.*

Proof. If there were p-torsion for $* < 2p-3$ in $H_*(B\Sigma_n;\mathbb{Z})$, by (3') the same would be true for the stable homology. But by a Sere class argument this contradicts Theorem 2.10. $\square$

3. Exercises

Exercise 3.1 (Recognizing BG). One way to recognize that a topological space is weakly homotopy equivalent to BG is to exhibit it as a quotient of a contractible topological space by G acting freely and properly discontinuously. In this exercise we use this to prove $|N(* /\!\!/ G)|$ arises this way.

(i) Consider the groupoid $G /\!\!/ G$ given by:
- objects given by G,
- morphisms given by $h \xrightarrow{g} gh$ for $g \in G$,
- composition given by multiplication.

Prove that $|N_*(G /\!\!/ G)|$ is contractible by using the fact that if $\eta\colon F \Rightarrow F'$ is a natural transformation, then $|NF|$ and $|NF'|$ are homotopic. (Hint: construct a natural transformation from the identity functor on $G /\!\!/ G$ to a constant functor.)

(ii) Multiplication on the right gives an action of G on $G /\!\!/ G$ and hence on $|N(G /\!\!/ G)|$. Prove that this action is free, properly discontinuous, and that $|N(G /\!\!/ G)|/G \cong |N(* /\!\!/ G)|$.

Exercise 3.2 (Group homology). The definition of group homology of G with coefficients in a $\mathbb{Z}[G]$-module M is $H_*(G; M) := \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M)$. That is, it is computed by the homology of $P_* \otimes_{\mathbb{Z}[G]} M$ where $P_* \to \mathbb{Z}$ is a projective $\mathbb{Z}[G]$-module resolution.

(i) As the geometric realisation of a simplicial set, $|N(G /\!\!/ G)|$ has a canonical CW structure with a single cell for each non-degenerate simplex. Using the cellular chains on $|N(G/G)|$ to construct a free $\mathbb{Z}[G]$-module resolution of $\mathbb{Z}$.
(ii) Prove that $H_*(BG; \mathbb{Z}) \cong \mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, \mathbb{Z})$ and conclude that the homology of BG is the group homology of G with coefficients in $\mathbb{Z}$.
(iii) How do you obtain $\mathrm{Tor}_*^{\mathbb{Z}[G]}(\mathbb{Z}, M)$ in terms of BG?

Exercise 3.3 (Homology of $B\mathbb{Z}/2$). Use cellular homology of $|N(* /\!\!/ \mathbb{Z}/2)|$ to compute $H_*(B\mathbb{Z}/2; \mathbb{Z})$.

Exercise 3.4 (Transfer maps).

(i) Prove that $p\colon X \to B$ is a k-fold cover, there is a map $\tau\colon \widetilde{H}_*(B; \mathbb{Z}) \to \widetilde{H}_*(X; \mathbb{Z})$ so that $p_* \circ \tau = k \cdot \mathrm{id}$. (Hint: send a singular simplex $\Delta^n \to B$ to its k distinct lifts.)
(ii) Prove that if G is a discrete group and $H \subset G$ is a subgroup of finite index, there is a model of BH which is a $[G : H]$-fold cover of BG.
(iii) Conclude that Proposition 1.6 holds.

Exercise 3.5 (Homology of BD_3).

(i) Use the arguments of Exercise 3.4 to deduce that $\widetilde{H}_*(BD_3; \mathbb{Z})$ is annihilated by multiplication with 6.
(ii) Use universal coefficient theorems to show that Example 1.5 follows once one proves that
$$H^*(BD_3; \mathbb{Z}/2) \cong \mathbb{Z}/2[x] \qquad \text{and} \qquad H^*(BD_3; \mathbb{Z}/3) \cong \mathbb{Z}/3[y]$$
with $|x| = 1$ and $|y| = 2$.
(iii) Prove (ii) using the cohomological Serre spectral sequence for the fibration sequence $B\mathbb{Z}/3 \longrightarrow BD_3 \longrightarrow B\mathbb{Z}/2$ associated to the extension
$$1 \longrightarrow \mathbb{Z}/3 \longrightarrow D_3 \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$
with coefficients in $\mathbb{Z}/2$ and $\mathbb{Z}/3$. (Hint: this extension is a semi-direct product with the non-trivial element of $\mathbb{Z}/2$ acting on $\mathbb{Z}/3$ by multiplication with -1.)

Exercise 3.6 (Dold's lemma). We will prove [Dol62, Lemma 2].

(i) Suppose we have a sequence of abelian groups and homomorphisms
$$0 \xrightarrow{\sigma_0} A_0 \xrightarrow{\sigma_1} A_1 \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} A_n$$
and homomorphisms $\tau_{k,m}\colon A_m \to A_k$ for $k \leq m \leq n$ such that
(a) $\tau_{k,k} = \mathrm{id}$,
(b) $\tau_{k,m} \circ \sigma_m = \tau_{k,m-1} \pmod{\mathrm{im}(\sigma_k)}$ for $k < m$.
Prove by induction over m that the map
$$T_m\colon A_m \longrightarrow \bigoplus_{k \leq m} A_k/\mathrm{im}(\sigma_k)$$

with kth component given by $\mathrm{proj} \circ \tau_{m,k}$, is an isomorphism and σ_m has a left inverse. (Hint: the left inverse to σ_m will be $T_{m-1}^{-1} \circ p \circ T_m \circ \sigma_m$ where $p\colon \bigoplus_{k \leq m} A_k/\mathrm{im}(\sigma_k) \to \bigoplus_{k \leq m-1} A_k/\mathrm{im}(\sigma_k)$ is the projection onto the first $m-1$ summands.)

(ii) In the sequence

$$0 \xrightarrow{\sigma_0} H_d(B\Sigma_0;\mathbb{Z}) \xrightarrow{\sigma_1} H_d(B\Sigma_1;\mathbb{Z}) \xrightarrow{\sigma_2} \cdots \xrightarrow{\sigma_n} H_d(B\Sigma_n;\mathbb{Z}),$$

use transfers to construct maps $\tau_{k,m}\colon H_d(B\Sigma_m;\mathbb{Z}) \to H_d(B\Sigma_k;\mathbb{Z})$ satisfying (a) and (b). Conclude that σ is always injective.

Exercise 3.7 (Some stable homotopy groups of spheres)*.*

(i) Use the results in Section 2.1 to prove that $\pi_1(\mathbb{S}) = \mathbb{Z}/2$.
(ii) Can you also compute $\pi_2(\mathbb{S})$? (Hint: [Arl90].)

Exercise 3.8 (Using Serre's finiteness theorem)*.* Serre proved that $\pi_*(\mathbb{S})$ is finite for $* > 0$. Combine this with Corollary 2.6 and Exercise 3.6 to prove that the sequence $B\Sigma_0 \xrightarrow{\sigma} B\Sigma_1 \xrightarrow{\sigma} B\Sigma_2 \xrightarrow{\sigma} \cdots$ exhibits homological stability. (Hint: you will not be able to give an explicit range.)

Remark 3.9*.* See [McD75] for a similar qualitative argument for configuration spaces of manifolds.

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