HOMOLOGICAL STABILITY MINICOURSE
LECTURE 4: MORE SUBTLE STABILITY PHENOMENA

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ABSTRACT. In the fourth lecture of this minicourse, we discuss two novel stability phenomena: representation stability and higher-order homological stability.

In this last lecture, we look past homological stability to a pair of more subtle stability phenomena: representation stability and higher-order homological stability. In either case we will do so through an illustrative example, and will not attempt to explain the general theory.

Remark. Other topics of recent interest that we will not discuss is stability phenomena near the cohomological dimension, e.g. [CFP14], and stable stability, e.g. [GRW17].

1. Representation stability

Close cousins of the symmetric groups are the configuration spaces of unordered points in a Euclidean space $\mathbb{R}^d$. We shall give the definition for a general topological space X:

Definition 1.1. The *configuration space of n unordered points in X* is given by

$$C_n(X) \coloneqq \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}/\Sigma_n,$$

where the elements of Σ_n act by permuting the particles.

By construction this is the quotient by the symmetric group Σ_n of the *configuration space of n ordered points in X* given by

$$\mathrm{Conf}_n(X) \coloneqq \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j\}.$$

Lemma 1.2.
(i) $\mathrm{Conf}_n(\mathbb{R}^d)$ *is* $(d-2)$*-connected.*
(ii) There is a $(d-1)$*-connected map* $C_n(\mathbb{R}^d) \to B\Sigma_n$.

Proof. Part (ii) follows from part (i) by observing that the action of Σ_n on $\mathrm{Conf}_n(\mathbb{R}^d)$ is free and proper, so the quotient by Σ_n is a homotopy quotient (see Exercise 3.3 (i)). To prove (i) we prove more generally that $\mathrm{Conf}_n(\mathbb{R}^d \setminus \{k \text{ points}\})$ is $(d-2)$-connected, by induction over n. For $n = 1$, we have $\mathrm{Conf}_1(\mathbb{R}^d \setminus \{k \text{ points}\}) = \mathbb{R}^d \setminus \{k \text{ points}\} \simeq \vee_k S^{d-1}$. The induction step uses the fibration sequences

$$\mathrm{Conf}_{n-1}(\mathbb{R}^d \setminus \{k+1 \text{ points}\}) \longrightarrow \mathrm{Conf}_n(\mathbb{R}^d \setminus \{k \text{ points}\}) \longrightarrow \mathbb{R}^d \setminus \{k \text{ points}\}$$

for $k \geq 1$, with right map obtained by remembering only the location of the first point [FN62]. $\square$

There are stabilisation maps $\sigma\colon C_n(\mathbb{R}^d) \to C_{n+1}(\mathbb{R}^d)$ given informally by adding a new particle far away in the e_1-direction (see Exercise 3.3 (ii)), fitting into a homotopy-commutative diagram

$$\begin{array}{ccc} C_n(\mathbb{R}^d) & \xrightarrow{\sigma} & C_{n+1}(\mathbb{R}^d) \\ \downarrow & & \downarrow \\ B\Sigma_n & \xrightarrow{\sigma} & B\Sigma_{n+1}. \end{array}$$

As a consequence, the homological stability result for symmetric groups implies one for the spaces $C_n(\mathbb{R}^d)$ in degrees $* < d-1$. In fact, it is true in all degrees:

Theorem 1.3. *The sequence of spaces $C_0(\mathbb{R}^d) \xrightarrow{\sigma} C_1(\mathbb{R}^d) \xrightarrow{\sigma} \cdots$ exhibits homological stability. More precisely, the map*

$$\sigma_*\colon H_*(C_n(\mathbb{R}^d);\mathbb{Z}) \longrightarrow H_*(C_{n+1}(\mathbb{R}^d);\mathbb{Z})$$

is a surjection for $* \leq \frac{n}{2}$ and an isomorphism for $* \leq \frac{n-1}{2}$.

Remark 1.4. The stabilisation maps are always injective, see Exercise 3.5.

Remark 1.5. Homological stability for unordered configuration spaces and their variants is a well-studied topic. Here is an incomplete list of references: [McD75, Seg79, LS01, Ker05, Chu12, RW13, BM14, KM15, CP15, KMT16, KM16, EVW16, Knu17, Pal18]. In many cases it is possible to compute the homology of configuration spaces outright [CLM76, Nap03, FT05, Pet17, BG18, Sch19, Pet20, Pag20].

Is something similar true for the ordered configuration spaces $\mathrm{Conf}_n(\mathbb{R}^d)$? The answer is no, because in the lowest degree where the reduced homology can be non-trivial by Lemma 1.2 (i), we have

$$H_{d-1}(\mathrm{Conf}_n(\mathbb{R}^d);\mathbb{Q}) \cong \mathbb{Q}^{n-1}.$$

This computation is more easily interpreted when we recall that $\mathrm{Conf}_n(\mathbb{R}^d)$ is a topological space with Σ_n-action and we describe this homology group as a rational Σ_n-representation: it is the kernel of the augmentation $\epsilon\colon \mathbb{Q}[\underline{n}] \to \mathbb{Q}$ (i.e. the *reduced regular representation*). Thus, when we take into account the naturally present group actions, the homology *does* admit a uniform description for n sufficiently large (in fact for all n in this particular case). This notion of stability is known as *representation stability* [CF13, Far14].

Let us now give a precise statement for the rational cohomology of ordered configuration spaces of a manifold M. Doing so uses that the rational representations of Σ_n are classified by partitions of n (or equivalently Young diagrams with n boxes) [FH91, Chapter 4], which we list as decreasing sequences $(i_1,\ldots,i_k)$ of positive integers with $\sum i_k = n$. For example, the trivial representation is (n) and the reduced regular representation is $(n-1,1)$. We can stabilise partitions by adding 1 to the first entry, and can think of this as an operation taking isomorphism classes of rational Σ_n-representations to rational Σ_{n+1}-representations. It may be extended to all isomorphism classes of rational Σ_n-representations by first decomposing these into irreducibles. We say that a sequence $\{V_n\}_{n\geq 0}$ of representations of symmetric groups exhibits *multiplicity stability* if for n sufficiently larger V_{n+1} can be obtained from V_n by this operation. Church proved the cohomology of ordered configuration spaces of manifolds has this property (though he proved much more):

Theorem 1.6 ([Chu12]). *Suppose M is a finite type manifold of dimension ≥ 2. Then for all $d \geq 0$ the representations $\{H^d(\mathrm{Conf}_n(M);\mathbb{Q})\}_{n\geq 0}$ exhibit multiplicity stability.*

Remark 1.7. By the universal coefficient theorem and self-duality of rational representation of symmetric groups, we could also have phrased this in terms of homology. Here is a reason to prefer cohomology: since M may be closed there are no stabilisation maps adding a point far away, like we used for $\mathbb{R}^d$. Rather, in contrast with the case of unordered configuration spaces there are now maps which forget a point. On cohomology these induce maps $H^*(\mathrm{Conf}_{n-1}(M);\mathbb{Q}) \to H^*(\mathrm{Conf}_n(M);\mathbb{Q})$ resembling stabilisation maps.

In general, representation stability is more subtle because for other sequences of groups or more complicated coefficients it is not possible to classify all representations and give them consistent names. This occurs even in such a simple case as the homology of configuration spaces with *integer coefficients*. In general, one needs to study the representation theory of certain categories which combine the groups acting and (de)stabilisation maps, and phrase the representation stability in terms of finite generation or presentation. For configuration spaces, the relevant category is FI (objects are non-empty finite sets and morphisms are injective functions) and the relevant objects are functors $\mathsf{FI} \to \mathsf{Ab}$. These are called FI-*modules* and, assuming that all homology groups are finitely generated, the analogue of exhibiting homological stability is being finitely-generated as a FI-module. The collection $\{H^d(\mathrm{Conf}_n(M);\mathbb{Z})\}_{n\geq 0}$ forms an FI-module using the forgetful maps discussed in the previous remark. That this is a finitely-presented FI-module for each $d \geq 0$ is proven in [CEF15]. For a proof closer to the ones in the previous lectures, see [MW19, MW20].

Remark 1.8. You can interpret ordinary homological stability as representation stability with trivial group action. In this case, the categorical representation theory is that of the poset $(\mathbb{N}, \leq)$. The category of functors $(\mathbb{N}, \leq) \to \mathsf{Ab}$ is equivalent to the category of graded $\mathbb{Z}[x]$-modules, and a degreewise finitely-generated graded $\mathbb{Z}[x]$-module is finitely generated if and only if it is finitely presented (because $\mathbb{Z}[x]$ is a noetherian ring) if and only if it is eventually constant.

Example 1.9. There is a lot of literature on representation stability for ordered configuration spaces [AAB15, HR17, KM18, FW18, MW19, MW20, Ram20] and their variants [JRW19, Gad17, Bib18]. Here is an incomplete list of classifying spaces of groups which exhibit representation stability and equally incomplete references:

- Groups related to mapping class groups [JR11, JR15, JRMD18, JR19] and Torelli groups [BHD12, Pat18, MPW19].
- Groups related to linear groups [Put15, PS17, MPW19].
- Groups related to automorphism groups of free groups [DP17].

2. Higher-order homological stability

The mapping class group $\Gamma_{g,1}$ of a surface $\Sigma_{g,1}$ with genus g and one boundary component, is the group of isotopy classes of diffeomorphisms of $\Sigma_{g,1}$ fixing the boundary pointwise. This is closely related to algebraic geometry as $\mathcal{M}_{g,1}$, the orbifold moduli space of curves with marked point and non-zero tangent vector, is homotopy equivalent to $B\Gamma_{g,1}$. Taking the boundary connected sum of $\Sigma_{g,1}$ with $\Sigma_{1,1}$ yields a surface diffeomorphic to $\Sigma_{g+1,1}$ and extending a diffeomorphism of $\Sigma_{g,1}$ by the identity to $\Sigma_{1,1}$ thus gives a stabilisation map

$$\sigma\colon B\Gamma_{g,1} \longrightarrow B\Gamma_{g+1,1}.$$

We mentioned in passing in the previous lecture that this sequence of classifying spaces exhibits homological stability: the map

$$\sigma_*\colon H_*(B\Gamma_{g,1};\mathbb{Z}) \longrightarrow H_*(B\Gamma_{g+1,1};\mathbb{Z})$$

is a surjection for $* \leq \frac{2g}{3}$ and an isomorphism for $* \leq \frac{2g-2}{3}$. This is the optimal range, proven in [GKRW19], but a proof of a nearly optimal result is explained in [Wah13]. A similar result with worse range can be obtained from theorem of [RWW17] that we explained in the previous lecture. (In particular, we note for later use that the mapping class groups assemble to a braided monoidal groupoid.)

This result informally says that the homology groups $H_d(B\Gamma_{g,1};\mathbb{Z})$ are independent of g when g is sufficiently large. In analogy with calculus, we could think of this as a function which is eventually constant, or equivalently as a function whose first derivative eventually vanishes. Higher-order homological stability is then analogous to its higher derivatives eventually vanishing.

To convert this analogy into a precise statement, we observe that the role of the first derivative can be played by the relative homology groups

$$H_*(B\Gamma_{g+1,1}, B\Gamma_{g,1};\mathbb{Z}),$$

which of course depend on the maps σ (even though the notation unfortunately does not reflect this). Their eventual vanishing is equivalent to homological stability, as the long exact sequence of a pairs implies that the induced map $\sigma_*\colon H_*(B\Gamma_{g,1};\mathbb{Z}) \to H_*(B\Gamma_{g+1,1};\mathbb{Z})$ is a surjection for $* \leq d$ and an isomorphism for $* < d$ if and only if $H_*(B\Gamma_{g+1,1}, B\Gamma_{g,1};\mathbb{Z})$ for $* \leq d$.

To formulate higher-order homological stability, we need to construct higher-order stabilisation maps. This is not straightforward—in this example they will not be unique(!)—but it turns out that there are maps

$$\varphi_*\colon H_d(B\Gamma_{g,1}, B\Gamma_{g-1,1};\mathbb{Z}) \longrightarrow H_{d+2}(B\Gamma_{g+3,1}, B\Gamma_{g+2,1};\mathbb{Z}).$$

The following is *secondary homological stability* for mapping class groups:

Theorem 2.1 ([GKRW19])**.** *The maps φ_* are surjections for $d \leq \frac{3g}{4}$ and isomorphisms for $d \leq \frac{3g-4}{4}$.*

Remark 2.2. Sometimes the secondary stabilisation maps are both isomorphisms in a range and zero, and one obtains an improved homological stability range.

Example 2.3. Few higher-order stability results have been proven at the time of writing these notes: [GKRW18, GKRW19, MW19, MPP19, GKRW20, Ho20, Him21]. One could ask whether such higher-order stability phenomena exist in any of the examples where homological stability is known.

2.1. **The strategy for proving Theorem 2.1.** We gave the "usual" homological stability result for symmetric groups, and isolated the crucial ingredients: given a braided monoidal groupoid G with some mild properties and objects $A, X \in \mathsf{G}$, we extracted a semi-simplicial set $W_n(A,X)_\bullet$ of "destabilisations" and given that these are homologically highly connected, homological stability follows using a spectral sequence argument. Let me now give a brief outline of the proof of Theorem 2.1, as it contains some useful ideas.

The input for the argument as in [RWW17] is a symmetric monoidal, or more generally braided monoidal, groupoid G. This structure on G endows $|N\mathsf{G}|$ with the structure of an E_2-algebra. In homotopy theory algebras can be not only coherently associative (E_1) or coherently commutative (E_∞), but there are also intermediate notions of commutativity: for an E_k-algebra the space of multiplications is a $(k-1)$-sphere, see Example 2.5. For $k = 1$ this means it is disconnected—there is a left and a right multiplication, and these can be quite different—and for $k = \infty$ this means it is contractible—from a homotopical viewpoint there is just a single multiplication.

More precisely, these are encoded by the operads of little k-discs [May72]. An *operad* $\mathcal{O}$ has Σ_r-spaces $\mathcal{O}(r)$ of r-ary operations, with a unit $1 \in \mathcal{O}(1)$, and compositions $\mathcal{O}(r) \times \mathcal{O}(k_1) \times \cdots \times \mathcal{O}(k_r) \to \mathcal{O}(k_1 + \cdots + k_r)$. These have two satisfy suitable equivariance, unitality, and associativity axioms. An $\mathcal{O}$-algebra is a space A with maps

$$\mathcal{O}(r) \times A^r \longrightarrow A,$$

which you should think of as defining operations on r-tuples of elements in A, indexed by the points of $\mathcal{O}(r)$. These should similarly satisfy suitable equivariance, unitality, and associativity axioms.

For the little k-discs operad, $E_k(r)$ is the space of k-tuples of rectilinear embeddings $D^k \hookrightarrow D^k$ with disjoint interior, Σ_r permuting the discs in the domain. The element $1 \in E_k(1)$ is the identity embedding, and composition is given by composition of embeddings.

Example 2.4. E_1 is homotopy equivalent to the associative operad, whose r-ary operations are the linear orders of $\underline{r}$. Algebras over the latter are associative algebras. E_∞ is homotopy equivalent to commutative operad, whose r-ary operations are contractible. Algebras over the latter are commutative algebras.

Example 2.5. The 2-ary operations of the E_k-operad are encoded by the space $E_k(2)$. This is the space of two k-discs in a k-discs, and is homotopy equivalent to S^{k-1}.

Since the mapping class groups assemble to a braided monoidal groupoid G, $|N\mathsf{G}| \simeq \bigsqcup_{g\geq 0} B\Gamma_{g,1}$ admits the structure of an E_2-algebra (see Exercise 3.7 for a more geometric approach).

In the homological stability arguments we gave before, we used from this E_2-structure just the multiplication on the right by a point representing $X \in |N\mathsf{G}|$ and some coherence from the braiding. Theorem 2.1 will need the full E_2-algebra structure. This is because it will build an approximation $\mathbf{A}$ to the E_2-algebra $\mathbf{R} \coloneqq |N\mathsf{G}|$ from free E_2-algebras. The free E_2-algebra functor Free^{E_2} is the left adjoint to the forgetful functor from E_2-algebras to spaces, and the underlying space of its values are of the form

$$\mathrm{Free}^{E_2}(X) \simeq \bigsqcup_{r\geq 0} E_2(r) \times_{\Sigma_r} X^r.$$

The E_2-algebra $\mathbf{A}$ will be a good approximation in a sense relevant to Theorem 2.1: it captures the homological stability properties in a range. One should compare this to how a CW-approximation of a skeleton of a topological space captures its homology in a range. The proof of Theorem 2.1 has the following steps (see [GKRW19] for details):

(1) *Understand how many free E_2-algebras one needs to build $\mathbf{A}$ such that it is a good enough approximation.* This requires two inputs: an understanding of $H_d(B\Gamma_{g,1};\mathbb{Z})$ for low d and g, and a connectivity result for certain semi-simplicial set. Unlike the semi-simplicial set $W_n(A,X)_\bullet$, these *E_1-splitting semi-simplicial sets* $S^{E_1}(g)_\bullet$

will be given by decompositions of a surface into boundary connected summands of lower genus.

(2) *Build the small cellular E_2-algebra approximation* $\mathbf{A}$. This uses techniques similar to CW-approximation for spaces, but in the category of E_2-algebras; it will only have three "E_2-cells." These are pushouts of the form

$$\begin{tikzcd} \mathrm{Free}^{E_2}(S^{k-1}) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Free}^{E_2}(D^k) \arrow[r] & Y \end{tikzcd}$$

in the category of E_2-algebras.

(3) *Construct the primary and secondary stabilisation maps.* The primary stabilisation maps will be σ as given above, and come from the E_2-algebra structure. However, the secondary stabilisation maps are constructed by obstruction theory.

(4) *Prove Theorem 2.1.* In steps (1) and (2), we made sure that $\mathbf{A}$ has the same homological stability properties as $\mathbf{R}$. In particular, $\mathbf{R}$ has (secondary) homological stability in a range when $\mathbf{A}$ does. This reduces the proof to a computation in $\mathbf{A}$. This is doable because F. Cohen completely computed the homology of free E_2-algebras in terms of certain homology operations [CLM76].

As a slogan, we call this a "multiplicative" approach to homological stability rather than an "additive" one, because it uses the full E_2-algebra structure instead of just the stabilisation maps extracted from it. In [GKRW18, GKRW20] we applied the same techniques to general linear groups.

3. Exercises

Exercise 3.1 (Interpretations of $C_n(\mathbb{C})$).

(i) Prove that $C_n(\mathbb{C})$ is an Eilenberg–Mac Lane space. Its fundamental group is the *nth braid group* Br_n, so Theorem 1.3 says that braid groups exhibit homological stability.

(ii) Prove that $C_n(\mathbb{C})$ is homeomorphic to the space of monic polynomials of degree $\leq n$ with complex coefficients and distinct roots.

Remark 3.2. Exercise 3.1 (ii) led to an interesting application of the homology of configuration spaces to the complexity of root-finding algorithms [Sma87].

Exercise 3.3 (Maps between configuration spaces).

(i) Construct a map $C_n(\mathbb{R}^d) \to C_n(\mathbb{R}^{d+1})$ and prove that $\mathrm{colim}_{d\to\infty} C_n(\mathbb{R}^d) \simeq B\Sigma_n$.

(ii) Construct a map $\sigma\colon C_n(\mathbb{R}^d) \to C_{n+1}(\mathbb{R}^d)$ fitting in a homotopy-commutative diagram

$$\begin{tikzcd} C_n(\mathbb{R}^d) \arrow[r, "\sigma"] \arrow[d] & C_{n+1}(\mathbb{R}^d) \arrow[d] \\ B\Sigma_n \arrow[r, "\sigma"] & B\Sigma_{n+1}. \end{tikzcd}$$

(iii) What conditions do we need to impose on a manifold M to be able to construct a similar map $\sigma\colon C_n(M) \to C_{n+1}(M)$?

Exercise 3.4 (Abelianisation of braid groups). Use Exercise 3.1 (i) to compute the abelianisations of the braid groups.

Exercise 3.5 (Applying Dold's Lemma to unordered configuration spaces).

(i) Construct "transfer maps" $\tau_{n,k} \colon H_*(C_n(\mathbb{R}^d);\mathbb{Z}) \to H_*(C_{n-k}(\mathbb{R}^d);\mathbb{Z})$ by summing over all ways of deleting k of the points.
(ii) Apply Dold's Lemma from the first lecture to prove that the stabilisation maps $\sigma_* \colon H_*(C_n(\mathbb{R}^d);\mathbb{Z}) \to H_*(C_{n+1}(\mathbb{R}^d);\mathbb{Z})$ are injective.
(iii) Generalise these results to any connected open manifold M replacing $\mathbb{R}^d$.

Exercise 3.6 (Betti numbers of configuration spaces of closed manifolds).

(i) Use the fibration sequence
$$\mathrm{Conf}_n(M) \longrightarrow C_n(M) \longrightarrow B\Sigma_n$$
to establish an isomorphism $H^*(C_n(M);\mathbb{Q}) \cong H^*(\mathrm{Conf}_n(M);\mathbb{Q})^{\Sigma_n}$.
(ii) Deduce from Theorem 1.6 that for any connected finite type manifold M of dimension ≥ 2, the ith rational Betti number of $C_n(M)$ is independent of n for $n \gg 0$.

Exercise 3.7. Let $\mathcal{M}(\Sigma_{g,1})$ be the colimit as $n \to \infty$ of the space of surfaces in $D^2 \times \mathbb{R}^n$ which are diffeomorphic to $\Sigma_{g,1}$ and coincide with $D^2 \times \{0\}$ near $\partial D^2 \times \mathbb{R}^n$. (For the topology on this space, see [GRW10]). It is a fact that
$$B\Gamma_{g,1} \simeq \mathcal{M}(\Sigma_{g,1}).$$
Endow $\bigsqcup_{g\geq 0} \mathcal{M}(\Sigma_{g,1})$ with the structure of an E_2-algebra.

Exercise 3.8 (The E_2-operad and braid groups).

(i) Prove that $E_2(r) \simeq \mathrm{Conf}_2(r)$, and hence an Eilenberg–Mac Lane space. Its fundamental group is the *rth pure braid group*, which is the kernel of the permutation homomorphism $\mathrm{Br}_n \to \Sigma_n$ recording how the strands of a braid permute the endpoints.
(ii) Prove that the free E_2-algebra $\mathrm{Free}^{E_2}(*)$ is homotopy equivalent to $\bigsqcup_{r\geq 0} C_n(r)$.
(iii) Compute $H_1(\mathrm{Free}^{E_2}(*);\mathbb{Z})$ using Exercise 3.4.

Exercise 3.9 (Stabilisation maps which are never isomorphisms). With rational coefficients the ranges in Theorem 2.1 can be improved to a surjection for $d \leq \frac{4g+1}{5}$ and an isomorphism for $d \leq \frac{4g-4}{5}$. It is a fact that $H_4(B\Gamma_{6,1}, B\Gamma_{5,1};\mathbb{Q}) \neq 0$ (using relations in the tautological ring). Prove that $H_{4+2k}(B\Gamma_{6+3k,1}, B\Gamma_{5+3k,1};\mathbb{Q})$ is non-zero for all $k \geq 0$.

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