eCHT. The $\mathbb{R}$-motivic Steenrod subalgebra $\mathcal{A}(1)$ and its 128 module structures.

Joint work with Prasit Bhattacharya and Bert Guillou.

$H^*(\_;\mathbb{F}_2)$ cohomology operations form Steenrod algebra $\mathcal{A}$.

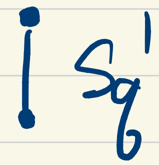
$H^*(X)$ is an $\mathcal{A}$-module.

$H^*(X) \overset{\cong}{?} M$ as an $\mathcal{A}$-module? realization problem.

X is a realization of M.

Notation: $S \xrightarrow{2} S \to S/2$ cofiber.

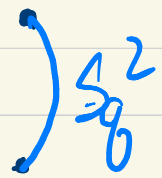
$H^*(S/2)$:



2 is detected by Sq^1

$\Sigma^1 S \xrightarrow{\eta} S \to S/\eta$

$H^*(S/\eta)$:



$Y := S/2 \wedge S/\eta$.

$H^*(Y)$:

Self map of $\Sigma^i X \xrightarrow{v} X$ gives families in $\pi_*(S^0)$.

by $S^{ik} \to \Sigma^{ik} \xrightarrow{v^k} X \to S^t$

[Adams], $\Sigma^8 S/_2 \xrightarrow{v_1^4} S/_2$ periodicity lower is better

[Davis-Mahowald 1980]

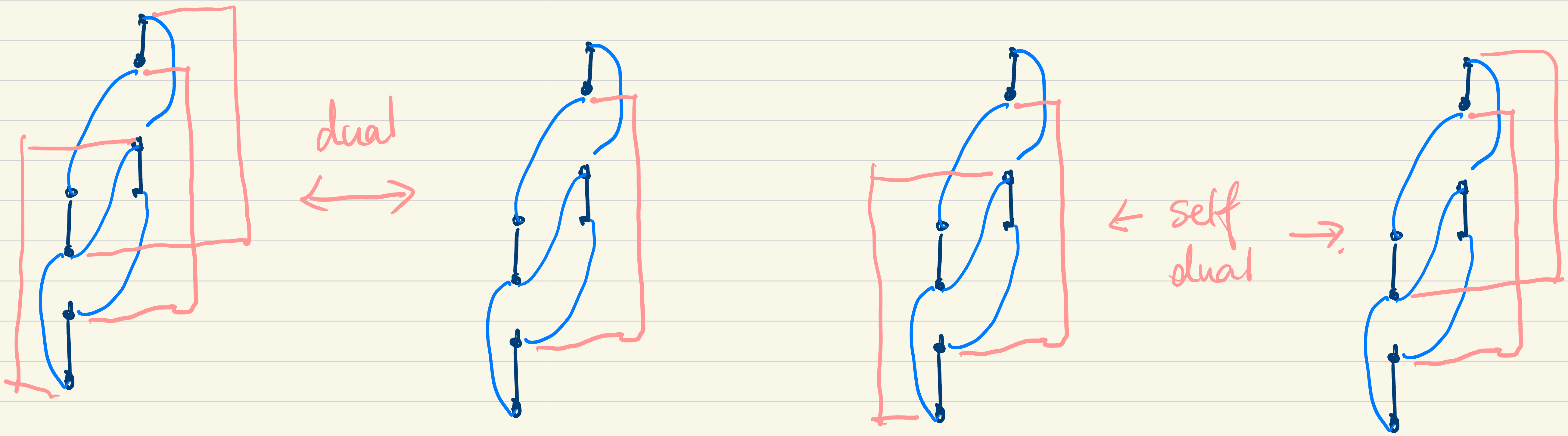
first construct A_1 using stunted real projective space.

$$P_5^8 \xrightarrow{g} P_3^6$$

$Y \xrightarrow{j} A_1$ induces surj in H^*, and cofiber$(j) \simeq \Sigma^3 Y$.

$$Y \to A_1 \to \boxed{\Sigma^3 Y \to \Sigma^1 Y}$$

$A(1)$. is gen by Sq^1 and Sq^2.



Thm [DM 1980]. $\exists$ 8 distinct v_1 selfmaps whose cofibers A_1 have 4 distinct htpy types, such that $H^*(A_1) \cong A(1)$.

Summary:
- construct A_1 such that $H^*(A_1) \cong A(1)$, as $A(1)$-mod.
- find v_1-selfmap on Y and identify $H^*(C(v_1))$ with $H^*(A_1)$
- analyze A-mod struture of $A(1)$. (htpy types of A_1).
- connect to v_1-selfmap: given A-mod str, it is realized by which v_1-self maps cofiber.

[Voevodsky 2003]. gives notion of motivic homotopy theory.

$S^{1,0} = \partial \Delta^1$ simplicial sphere.

$S^{1,1} = \mathbb{G}_m$ geometric sphere.

$SH^{\mathbb{R}} \xrightarrow{Be} SH^{C_2} \xrightarrow{res} SH$

no action $S \xrightarrow{2} S$ $\qquad$ $S \xrightarrow{2} S$

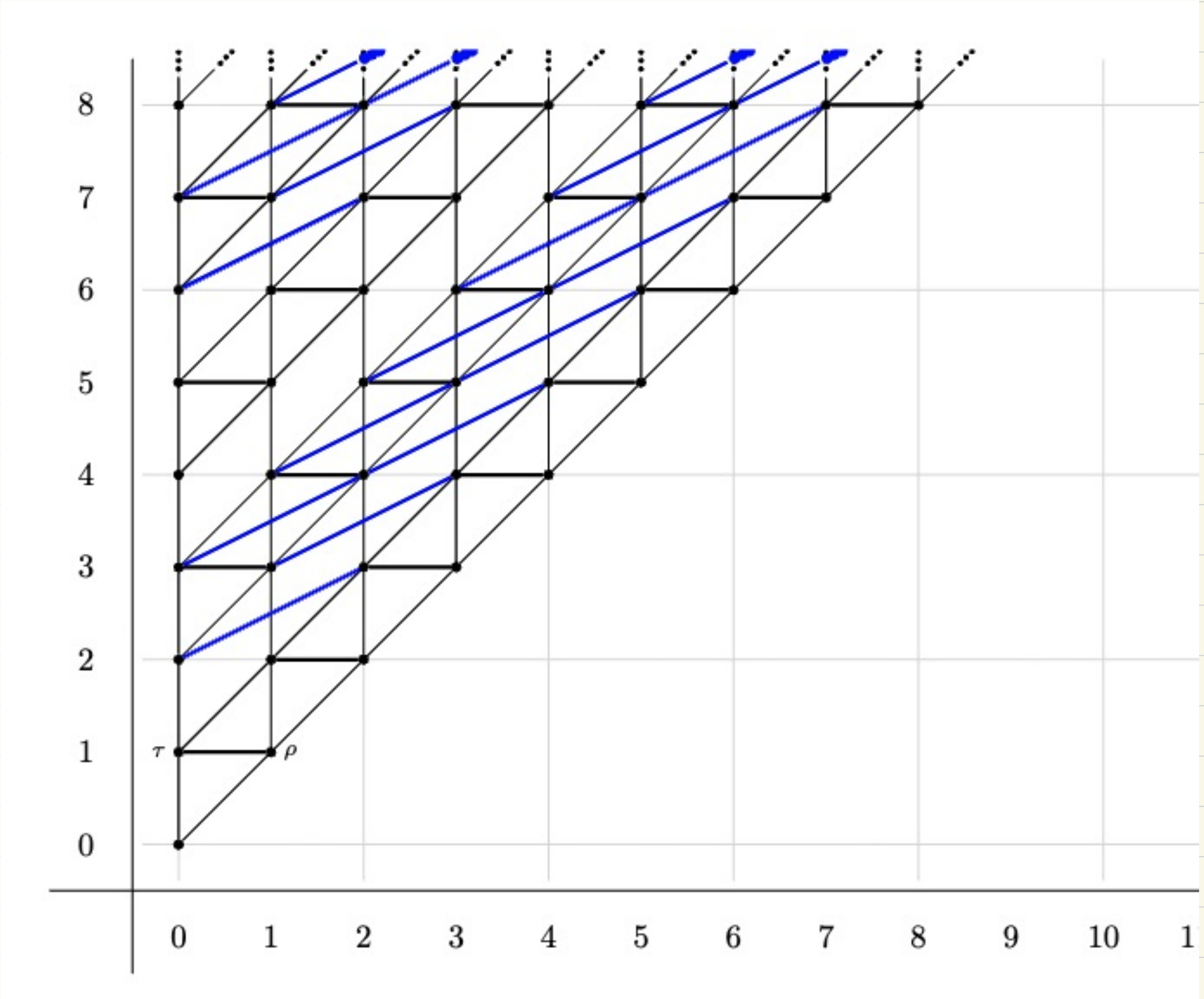
$S \xrightarrow{h} S$

$\Sigma^{1,1} S \xrightarrow{\eta} S$ $\qquad$ $S \xrightarrow{\eta} S$.

we have Steenrod algebra $A^{\mathbb{R}}$ that is a Hopf coalgebroid.

$h = 1 - \varepsilon$. $\varepsilon: S^{1,1} \wedge S^{1,1} \to S^{1,1} \wedge S^{1,1}$ is the $\mathbb{R}$-motivic Hopf map.

η is the Hopf construction of $S^{1,1} \times S^{1,1} \xrightarrow{m} S^{1,1}$.

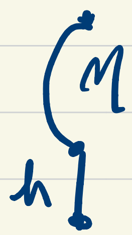


$H^{**}(pt) = \mathbb{M}_2 = \mathbb{F}_2[\tau, \rho].$

$\rho : S^0 \to S^\sigma$

$Sq^1 \tau = \rho$

we construct $A_1^{\mathbb{R}}$

$Q:$ 

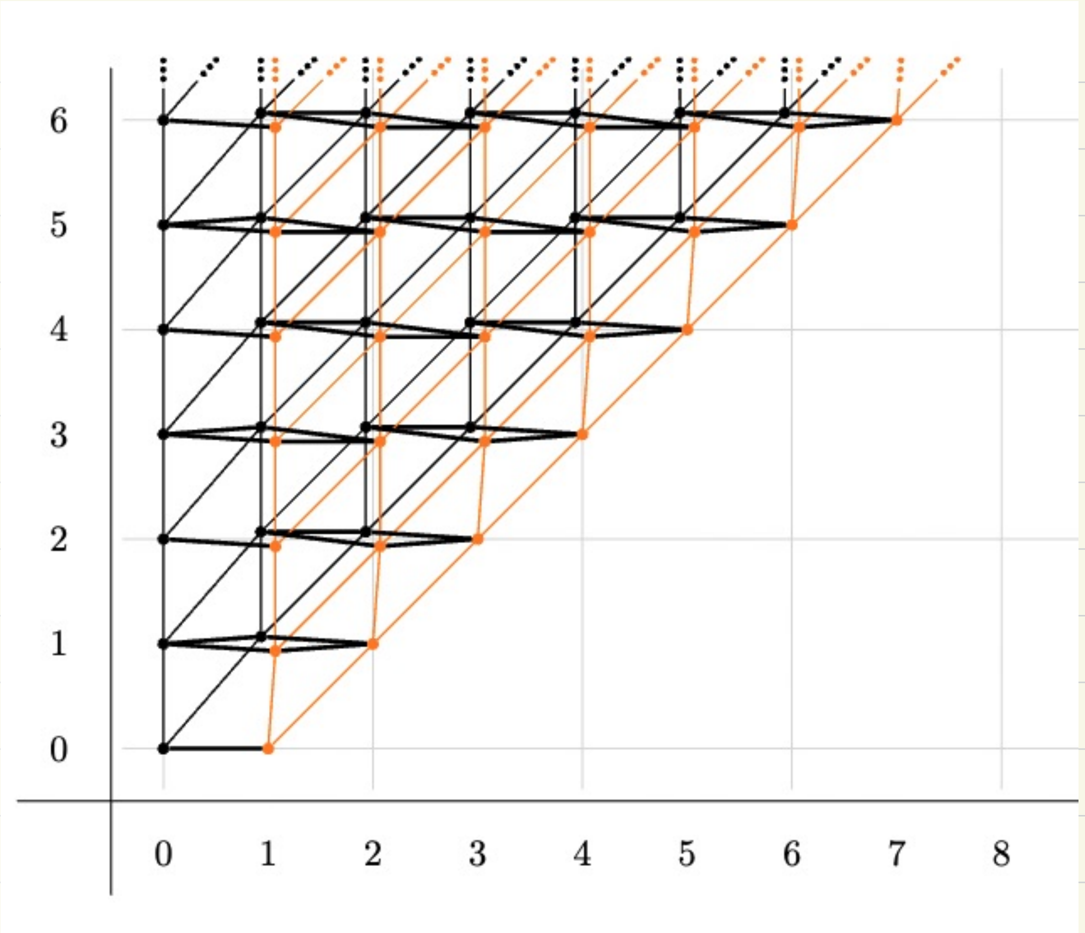
$e(H^{**}(Q)^{\otimes 3})$ is a free $A(1)$-mod.

we compute v_1 in

$$\mathrm{Ext}_{A^{\mathbb{R}}}(H^{**}(Y \wedge DY), \mathbb{M}_2^{\mathbb{R}}).$$
$$\Downarrow$$
$$[Y, Y].$$

identify $H^{**}(C(v_1)) \cong H^{**}(A_1^{\mathbb{R}})$.

we don't know if $C(v_1)$ has the same htpy type of A_1.



$A(0)^{\mathbb{R}}$. as an $A(0)^{\mathbb{R}}$

$S/2$ and S/h are two realizations of $A(0)^{\mathbb{R}}$.

$SH^{C_2} \xrightarrow{\Phi} SH$. inverts ρ.

$\Phi(\underline{H\mathbb{F}_2}) = H\mathbb{F}_2[t]. \simeq \bigvee_i \Sigma^i H\mathbb{F}_2$,

$\Phi(2) = 2$, $\Phi(h) = \emptyset$.

$\Phi(\underline{Sq}^2 H^{**}(X)) = Sq^1(\underline{\Phi} H^{**}(X))$,

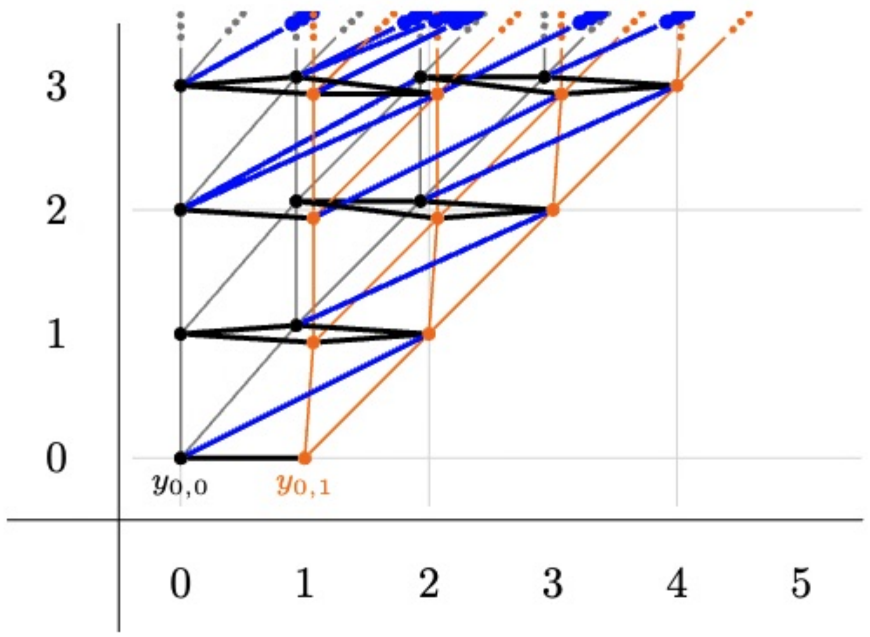


Figure: $H^{**}(S/2)$

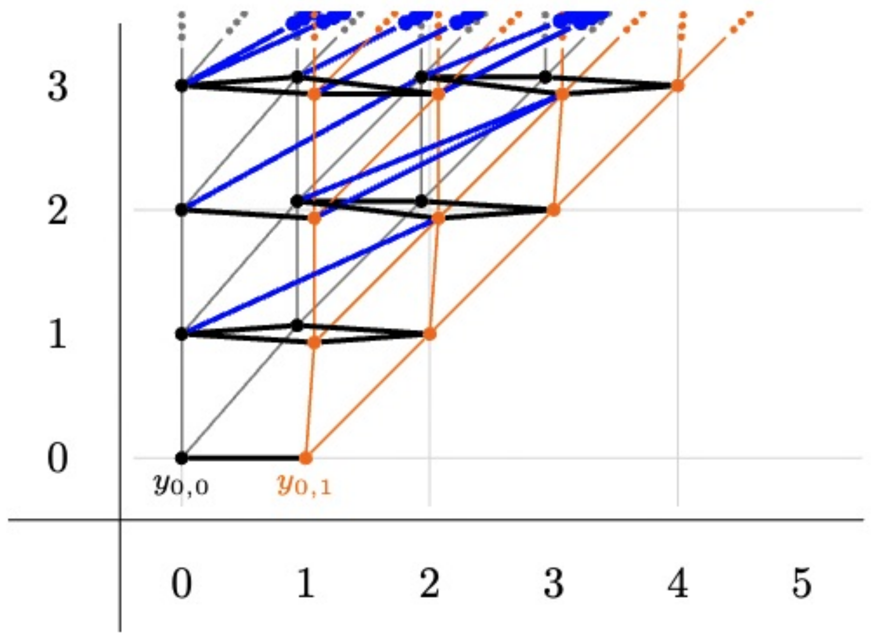
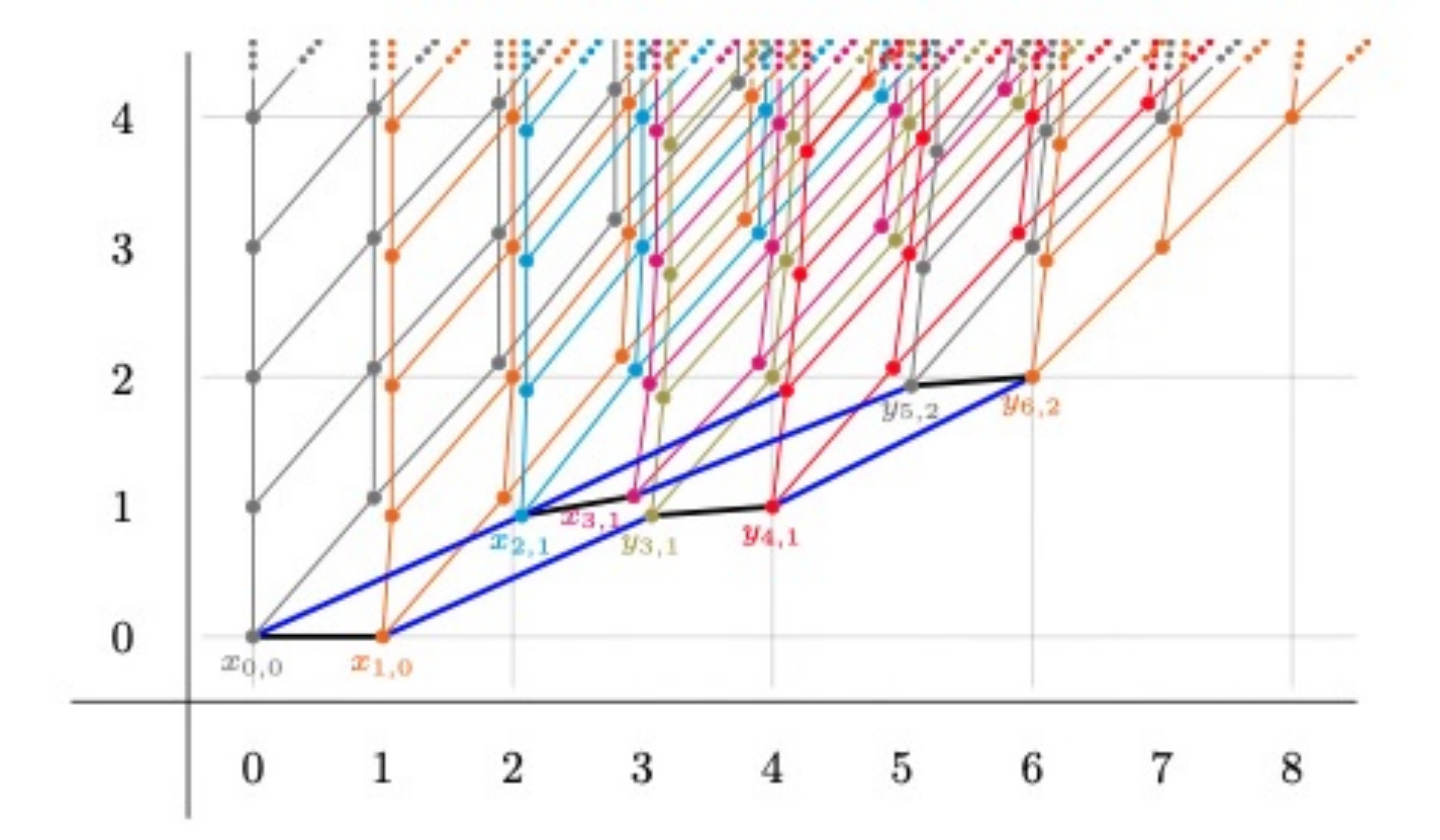


Figure: $H^{**}(S/h)$



$H^{**}(kgl) \cong A^{\mathbb{R}} /\!/ A(1)^{\mathbb{R}}$

if treat each cone as dot.

$Sq^4 x_{00} = \beta_{03}(\rho \cdot y_{31}) + (1+\beta_{03}+\beta_{14})(\tau \cdot y_{41}) + \alpha_{03}(\rho \cdot x_{31})$

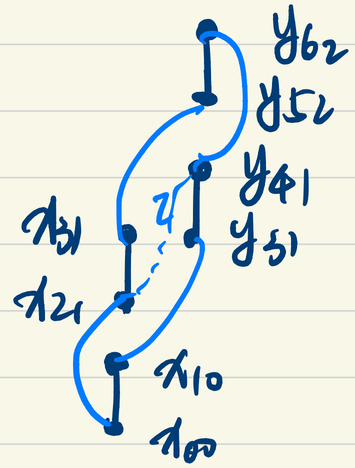
$Sq^4 x_{10} = y_{52} + \beta_{14}(\rho \cdot y_{41})$

$Sq^4 x_{21} = \beta_{26}(\tau \cdot y_{62}) + \beta_{25}(\rho \cdot y_{52}) + j_{24}(\rho^2 \cdot y_{41})$

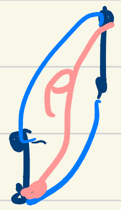
$Sq^4 x_{31} = (\beta_{25}+\beta_{26})(\rho \cdot y_{62})$

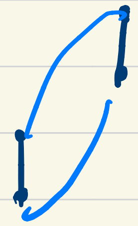
$Sq^4 y_{31} = \gamma_{36}(\rho \cdot y_{62})$ with $j_{24} = \beta_{03}\gamma_{36} + \alpha_{03}(\beta_{25}+\beta_{26})$

$Sq^8 x_{00} = \beta_{06}(\rho^2 \cdot y_{62})$



in $SH^{\mathbb{R}}$, we have $Y_2 := S/2 \wedge S/\eta$ $\quad$ $Y_h := S/h \wedge S/\eta$.

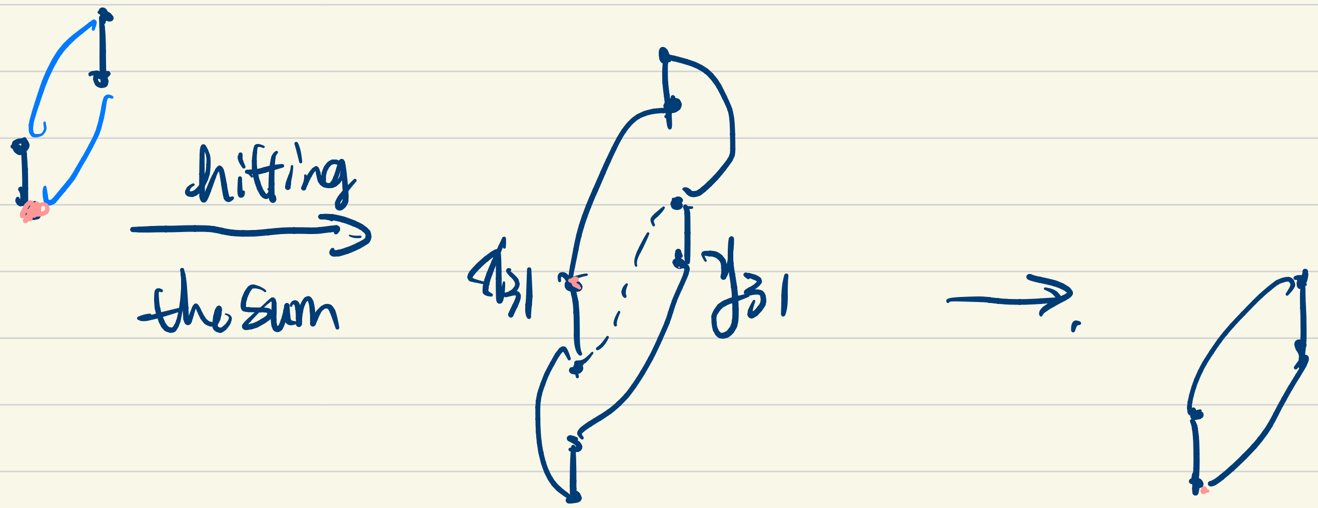




v_1-self map is between $\{Y_2, Y_h\}$. !

Thm [BGL, 2021].

$$\Sigma^{3,1} H^{**}(Y_\varepsilon) \to A(1)^{\mathbb{R}} \to H^{**}(Y_\delta).$$



Let $\varepsilon = \begin{cases} h & \text{if } \beta_{25} + \beta_{26} + \gamma_{36} = 0 \\ 2 & \cdots\ 1 \end{cases}$ $\qquad$ $\delta = \begin{cases} h & \text{if } \alpha_{03} + \beta_{03} = 0 \\ 2 & \quad 1 \end{cases}$