Computations of height 2 higher real K-theory spectra at prime 2.

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$E_2^{hQ_8}$

stable homotopy group of spheres $\pi_*^{st} S^0$

n	1	2	3	4	5	6
	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/8 \oplus \mathbb{Z}/3$	0	0	$\mathbb{Z}/2$

n	7	8	9	10	11
	$\mathbb{Z}/16 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/5$	$(\mathbb{Z}/2)^2$	$(\mathbb{Z}/2)^3$	$\mathbb{Z}/2 \oplus \mathbb{Z}/3$	$\mathbb{Z}/8 \oplus \mathbb{Z}/9 \oplus \mathbb{Z}/7$

n	12	13	14	15
	0	$\mathbb{Z}/3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/2 \oplus \mathbb{Z}/32 \oplus \mathbb{Z}/3 \oplus \mathbb{Z}/5$

For each height $n \geq 1$, $\exists$ "v_n-periodic" families.

The height is related to formal group laws, seen in the structure of ANSS.

$\uparrow$

filtration by heights

Morava K-theory: at prime p

$K(n)$

$\pi_* K(n) = \mathbb{F}_p[v_n^{\pm 1}]$ $|v_n| = 2(p^n-1)$.

$K(0) = H\mathbb{Q}$ $K(\infty) = H\mathbb{F}_p$

X: finite p-local spectrum,

$K(n)_* X = 0 \to K(n-1)_* X = 0$.

More precisely, we can construct Bousfield localization functors L_n

($E(n)$: Johnson-Wilson thy)

$$X \to \cdots \to L_n X \to L_{n-1} X \to \cdots \to L_1 X \to L_0 X = H\mathbb{Q} \wedge X.$$

Thm. $\varprojlim L_n X \simeq X$.

Take fiber of

$$\begin{tikzcd} L_n X \arrow[r] & L_{n-1} X \\ M_n X \arrow[u] & \end{tikzcd}$$

monochromatic layers.

Focus on $X = S^0_{(p)}$.

Chromatic fracture square.

$$\begin{tikzcd} L_n X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L_{K(n)} X \arrow[d] \\ L_{n-1} X \arrow[r] & L_{n-1} L_{K(n)} X \end{tikzcd}$$

Study $K(n)$-local spheres $L_{K(n)} S^0$.

$L_{K(n)}S^0$:

Construct Morava E-theories E_n (Lubin-Tate theories).

$E_n \circlearrowleft S_n$. small Morava stabiliser group.

$E_n \circlearrowleft \mathbb{G}_n$ Morava stabiliser group.

Devinatz-Hopkins showed that $E_n^{h\mathbb{G}_n} \simeq L_{K(n)}S^0$.

$$E_2^{s,t} = H_c^s(\mathbb{G}_n, \pi_t E_n) \Rightarrow \pi_{t-s} E_n^{h\mathbb{G}_n}$$

We can approximate $E_n^{h\mathbb{G}_n}$ by E_n^{hG} where G is a finite subgroup of $\mathbb{G}_n$.

"E_n^{hG}": Bauer, Hill-Hopkins-Ravenel, Beaudry-Bobkova-Hill-Stojanoska, Hill-Shi-Wang-Xu, Goerss-Henn-Mahowald-Rezk, ...

Homotopy fixed points spectral sequence.

X. w/ an action by G.

$$F(EG_+, X) = X^{hG}$$

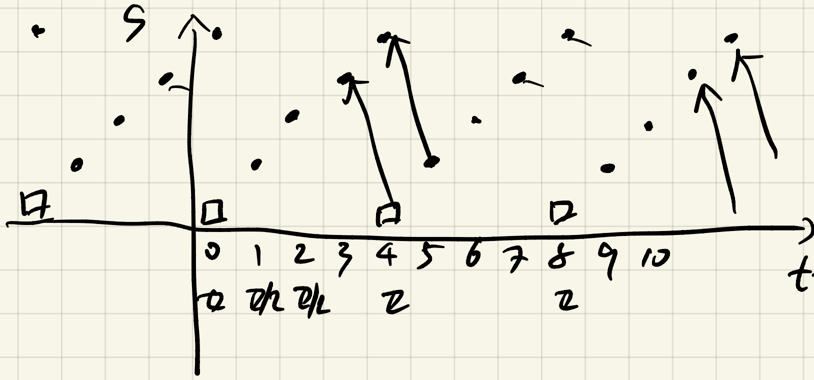
$$E_2^{s,t} = H^s(G, \pi_t X) \Rightarrow \pi_{t-s} X^{hG}.$$

<u>example</u>. KU^{hC_2}

$$\pi_* KU = \mathbb{Z}[u^{\pm 1}] \quad |u| = 2$$

C_2 action on $\pi_* KU$. $u \longmapsto -u$ at degree 2

$E_2^{s,t} = H^s(C_2, \pi_t KU)$ $\qquad \pi_t KU = \begin{cases} \mathbb{Z} & t \equiv 0 \pmod 4 \\ \mathbb{Z}_\sigma & t \equiv 2 \pmod 4 \end{cases}$



$t \equiv 0 \pmod 4$

$t-s$ $\quad \square = \mathbb{Z}$

$\bullet : \mathbb{Z}/2$

d_3 differentials in HFPSS

$KU^{hC_2} = KO$.

$p = 2 \qquad n = 2$

finite subgroup for $\mathbb{G}_2$ is well understood.

maximal: G_{48}

$$G_{24} \cap \mathbb{S}_2 = G_{24} \cong Q_8 \rtimes C_3$$

$E_2^{hQ_8}$

$E_2^{hG_{48}} \simeq L_{K(2)} Tmf$

Upgrade: $\underline{H}^s(Q_8, \pi_\star E_2) \Rightarrow \pi_{\star - s} E_2^{hQ_8}$.

Q_8 rep. $1, \sigma_i, \sigma_j, \sigma_k, \mathbb{H}$.

Q_8 action on $\pi_* E_2$

$$\omega_*(v_1) = v_1, \qquad \omega_*(u^{-1}) = \zeta^2 u^{-1},$$
$$i_*(u^{-1}) = \frac{v_1 - u^{-1}}{\zeta^2 - \zeta}, \qquad i_*(v_1) = \frac{v_1 + 2u^{-1}}{\zeta^2 - \zeta},$$
$$j_*(u^{-1}) = \frac{\zeta v_1 - u^{-1}}{\zeta^2 - \zeta}, \qquad j_*(v_1) = \frac{v_1 + 2\zeta^2 u^{-1}}{\zeta^2 - \zeta},$$
$$k_*(u^{-1}) = \frac{\zeta^2 v_1 - u^{-1}}{\zeta^2 - \zeta}, \qquad k_*(v_1) = \frac{v_1 + 2\zeta u^{-1}}{\zeta^2 - \zeta}.$$

Beaudry: Towards the homotopy of the $K(2)$-local Moore spectrum at $p=2$. based on notes of Henn.

$$\pi_* E_2 = \mathbb{W}[[v_1]][u^{\pm 1}] \qquad |v_1| = 0, \ |u| = 2$$

Use the same technique to calculate the E_2-page Q_8-HFpss(E_2).

Structure of the E_2 page:

(1). $G_{24} \cong Q_8 \rtimes C_3$.

$$\begin{tikzcd} E_2^{hG_{24}} \arrow[r, "res"] \arrow[rr, bend right, "\times 3"'] & E_2^{Q_8} \arrow[r, "tr"] & E_2^{hG_{24}} \end{tikzcd} \quad \text{an equivalence}$$

(2). norm structures in HFpss. (HHR).

roughly speaking.

$d_r(x) = y$ in $H \leq G$

predict some differential

$d_{(r-1)|G/H|+1}$.

(3) · $RO(G)$-grading.

C_4-HFPSS is well understood, fully calculated in HHR, BBHS.

norm up periodicities to get periods in Q_8-HFPSS.

Lemma 2.21. *The following permanent cycles in C_4-HFPSS(E_2) [HHR17][BBHS20] are periodic classes.*

- *The class $\bar{\mathfrak{d}}_1$ gives $(1+\sigma+\lambda)$-periodicity.*
- *The class $u_{8\lambda}$ gives $(16-8\lambda)$-periodicity.*
- *The class $u_{4\sigma}$ gives $(4-4\sigma)$-periodicity.*
- *The class $u_{4\lambda}u_{2\sigma}$ gives $(10-4\lambda-2\sigma)$-periodicity.*

Corollary 2.22. *We have the following $RO(Q_8)$-periodicities in Q_8-HFPSS(E_2).*

- $N_{C_4}^{Q_8}(\bar{\mathfrak{d}}_1)$:

$$1+\sigma_i+\sigma_j+\sigma_k+\mathbb{H}$$

- $N_{C_4}^{Q_8}(u_{4\sigma})$:

$$4+4\sigma_i-4\sigma_j-4\sigma_k$$
$$4+4\sigma_j-4\sigma_i-4\sigma_k$$
$$4+4\sigma_k-4\sigma_i-4\sigma_j$$

- $N_{C_4}^{Q_8}(u_{4\lambda}u_{2\sigma})$:

$$10+10\sigma_i-2\sigma_j-2\sigma_k-4\mathbb{H}$$
$$10+10\sigma_j-2\sigma_i-2\sigma_k-4\mathbb{H}$$
$$10+10\sigma_k-2\sigma_j-2\sigma_i-4\mathbb{H}$$

- $N_{C_4}^{Q_8}(u_{8\lambda})$:

$$16+16\sigma_i-8\mathbb{H}$$
$$16+16\sigma_j-8\mathbb{H}$$
$$16+16\sigma_k-8\mathbb{H}$$

$\Rightarrow$ Q_8 HFPSS is 64-periodic.

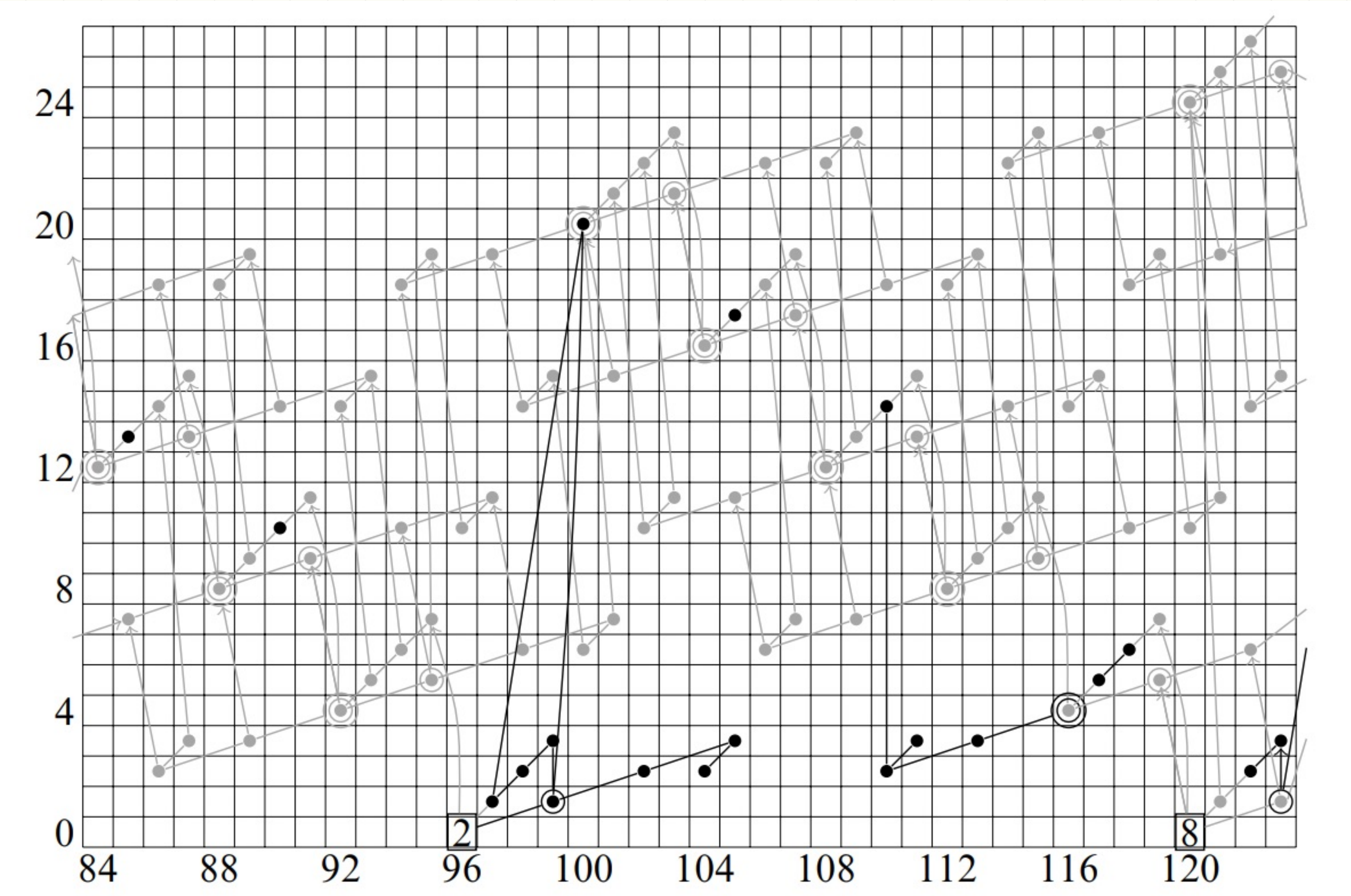
period given by a class D^8

Q_8 HFPSS splits as three copies. there are no differentials between different copies.

Vanishing line argument.

norm structures

Thm (DLS). E_{4k+2}, Q_8-HFPSS has a strong vanishing line at degree $\underline{2^{k+5}-9}$



E_2: $s = 23$

Bauer: 6-fold Toda bracket.

Use horizontal vanishing line. d_5, d_{13}, d_{23}

example. hidden extension.

HHR: G: cyclic 2-group.

$G' \triangleleft G$, index 2.

Then in $\pi_\star(F(EG_+, X))$,

- $\ker(\mathrm{res}^G_{G'}) = \mathrm{im}\, a_\sigma$
- $\mathrm{im}(\mathrm{tr}^G_{G'}) = \ker a_\sigma$

BBHS: at stem 22, $\exists$ exotic restriction, exotic transfer.

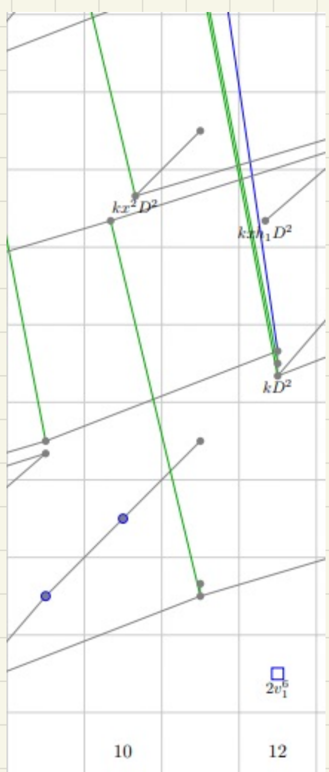
$\Rightarrow$ hidden 2 extension

Lem. At stem 54, there is a hidden 2 extension from $D^6 h_2^2$ to $k^2 D^8 x^2$

Cor. There is a hidden 2 extension from kDh_2^2 to $k^3x^2D^3$

Cor $d_5(Dh_2) = kDh_2^2$

$\Rightarrow d_{13}(2Dh_2) = k^3x^2D^3$



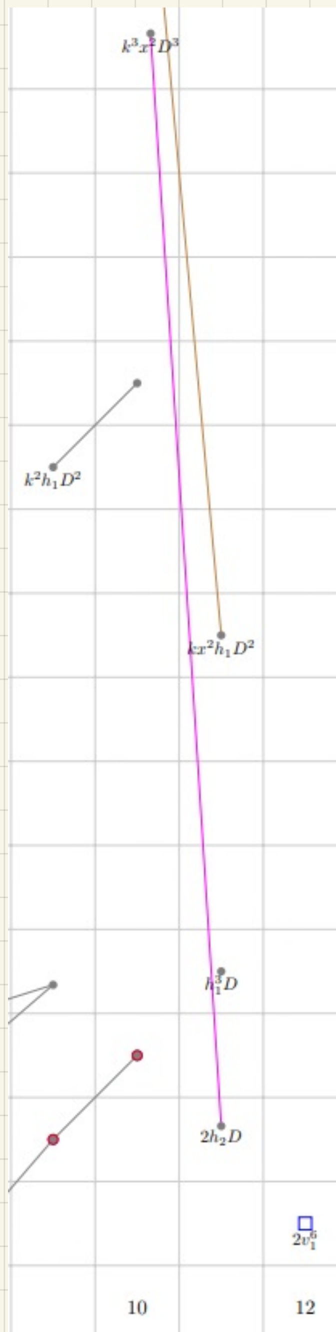


Table 8. HPFSS differentials, integer page

toprule (s,f)	x	r	$d_r(x)$	Proof
$(12,0)$	v_1^6	3	$v_1^4h_1^3$	Proposition 4.10 (restriction)
$(8,0)$	D	5	$D^{-2}gh_2$	Corollary 4.15 (vanishing line)
				or Proposition 4.41 (restriction)
$(8,0)$	$4D$	7	$D^{-2}gh_1^3$	Proposition 4.17 ($8\nu = \eta^3$)
$(16,0)$	$2D^2$	7	$D^{-1}gh_1^3$	Proposition 4.17
$(32,0)$	D^4	7	Dgh_1^3	Proposition 4.28 (vanishing line)
$(9,1)$	Dh_1	9	$D^{-5}g^2c$	Corollary 4.32
$(41,1)$	D^5h_1	9	$D^{-1}g^2c$	Corollary 4.32
$(16,2)$	Dc	9	$D^{-5}g^2dh_1$	Corollary 4.16
$(48,2)$	D^5c	9	$D^{-1}g^2dh_1$	Proposition 4.18
$(17,1)$	D^2h_1	9	$D^{-4}g^2c$	Proposition 4.38
$(49,1)$	D^6h_1	9	g^2c	Proposition 4.38
$(24,2)$	D^2c	9	$D^{-4}g^2dh_1$	Corollary 4.34
$(56,2)$	D^6c	9	g^2dh_1	Corollary 4.34
$(30,2)$	D^2d	11	$D^{-4}g^3h_1$	Proposition 4.30 (restriction)
$(62,2)$	D^6d	11	g^3h_1	Proposition 4.30 (restriction)
				or Proposition 4.42 (vanishing line)
$(23,3)$	Ddh_1	11	$D^{-5}g^3h_1^2$	Corollary 4.35
$(55,3)$	D^5dh_1	11	$D^{-1}g^3h_1^2$	Corollary 4.35
$(17,3)$	Dch_1	13	$2D^{-8}g^4$	Proposition 4.14 (vanishing line)
$(49,3)$	D^5ch_1	13	$2D^{-4}g^4$	Proposition 4.18 (transfer)
$(11,1)$	$2Dh_2$	13	$D^{-8}g^3d$	Proposition 4.25 (hidden 2 extension)
$(43,1)$	$2D^5h_2$	13	$D^{-4}g^3d$	Proposition 4.25
$(-7,1)$	$D^{-1}h_1$	23	$D^{-16}g^6$	Proposition 4.14 (vanishing line)
$(18,2)$	$D^2h_1^2$	23	$D^{-13}g^6h_1$	Corollary 4.22
$(43,3)$	$D^5h_1^3$	23	$D^{-10}g^6h_1^2$	Corollary 4.22