Higher Scissors Congruence

joint w/ Bohmann, Gerhardt, Merling, and Zakharevich

X any standard geometry $(E^n, H^n, \text{ or } S^n)$
Euclidean, hyperbolic, spherical

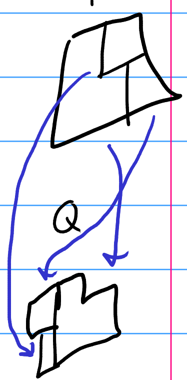
$G \leq \mathrm{Isom}(X)$ — no topology

Consider polytopes in X (bounded, non-convex)

Δ^n = convex hull of $(n+1)$ pts in general position

$P = \cup_i \Delta^n_i$ finite union

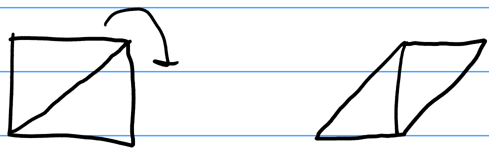
P, Q are <u>G-scissors congruent</u> if $\exists$ almost-disjoint covers:



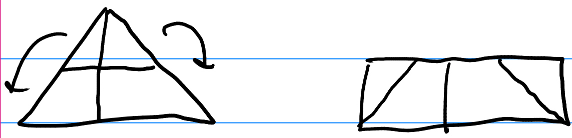
$$P = \cup_i P_i \qquad \mathrm{int}\, P_i \cap \mathrm{int}\, P_j = \emptyset$$
$$Q = \cup_i Q_i \qquad \mathrm{int}\, Q_i \cap \mathrm{int}\, Q_j = \emptyset$$

and $P_i \underset{\cong}{\xrightarrow{g_i}} Q_i$ $\qquad g_i \in G$

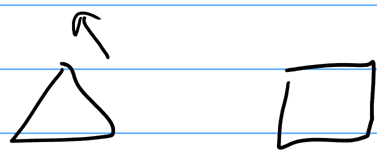
<u>Ex.</u>



usually $G = \mathrm{Isom}(X)$



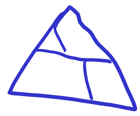
$G = T(n)$ on E^n
$\cong \mathbb{R}^n$



same area
not $T(2)$-s.c.

In E^2, $P \underset{sc}{\sim} Q \iff$ same area.

Scissors congruence group



$P = \bigcup_i P_i$, interiors disjoint

$$\mathcal{P}(X, G) = \mathbb{Z}^{\oplus \text{polytopes}} \Big/ \begin{array}{l} [P] = \sum_i [P_i] \\ \text{if } P_i \text{ are an a.d. cover} \\ [P] = [gP] \quad g \in G \end{array}$$

$$P \underset{G\text{-s.c.}}{\sim} Q \;\underset{\substack{\Longleftarrow \\ \text{if } G \text{ acts transitively}}}{\Longrightarrow}\; [P] = [Q] \text{ in } \mathcal{P}(X, G)$$

$$\mathcal{P}(E^1) = \mathbb{R} \quad (\text{length})$$

Hilbert's 3rd: Is $\mathcal{P}(E^3) \cong \mathbb{R}$?

$$\mathcal{P}(E^2) = \mathbb{R} \quad (\text{area})$$

$$\mathcal{P}(E^3) \not\hookrightarrow \underset{(\text{volume})}{\mathbb{R}} \oplus \underset{(\text{Dehn invariant}) \ (1901)}{(\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}/\pi\mathbb{Z})}$$

Dehn-Sydler Thm (Sydler 1965)

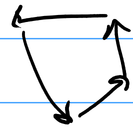
$$\sum_{e \in \text{edges}} \ell(e) \otimes \text{angle at } e$$

$$\mathcal{P}(E^4) \hookrightarrow \mathbb{R} \oplus (\mathbb{R} \otimes \mathbb{R}/\pi\mathbb{Z})$$

(Jessen 1968)

$$\mathcal{P}(E^5) = ?$$

$$\hookrightarrow \mathbb{R} \oplus \cdots \oplus \cdots$$



$$0 \to \mathcal{P}(E^2, T(2)) \to \underset{(\text{area})}{\mathbb{R}} \oplus \underset{\mathbb{R}P^1}{\bigoplus} \mathbb{R} \to \mathbb{R} \to 0$$

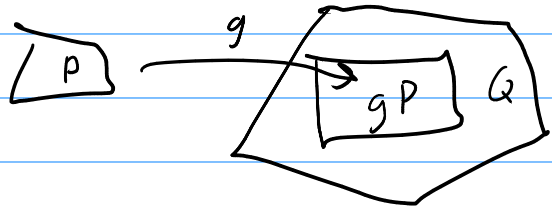
$T(2) \cong \mathbb{R}^2$ translation group

Categorify!

Polytopes form a category $\mathcal{P}_G^X$

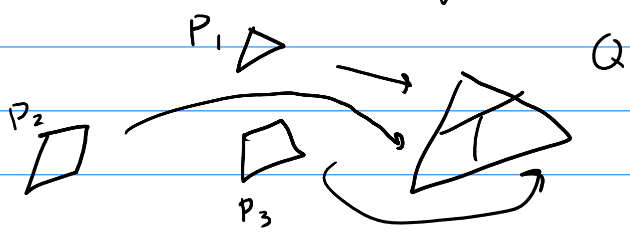
objects: polytopes $P \subseteq X$

morphisms: $P \xrightarrow{g} Q$ $\qquad g \in G$ s.t. $gP \subseteq Q$



A <u>cover</u> is a finite collection of maps in $\mathcal{P}_G^X$ $\{P_i \xrightarrow{g_i} Q\}_{i \in I}$ s.t.

$$Q = \bigcup_i g_i P_i, \quad \text{disjoint interiors}$$



Category of covers $\mathcal{W}(\mathcal{P}_G^X)$

objects: $\{P_i\}_{i \in I}$ finite formal disjoint unions

morphisms: $\{P_i\}_{i \in I} \longrightarrow \{Q_j\}_{j \in J}$

$$I \xrightarrow{f} J$$
$$P_i \xrightarrow{g_i} Q_{f(i)}$$

s.t. $\quad \{P_i \rightarrow Q_j\}_{i \in f^{-1}(j)}$ is a cover.

Note: $P \underset{G\text{-s.c.}}{\sim} Q \iff P, Q$ in same component of $\mathcal{W}(\mathcal{P}^X_G)$

<u>Def.</u> (Zakharevich) Scissors congruence K-theory is the group completion

$$K(\mathcal{P}^X_G) = \Omega B(B\mathcal{W}(\mathcal{P}^X_G))$$

w.r.t. disjoint union.

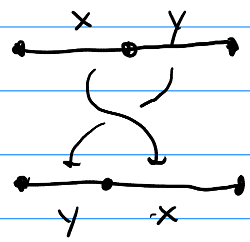
$$K_0(\mathcal{P}^X_G) = \overbrace{\mathcal{P}(X,G)}^{\text{classical s.c. group}} = \mathbb{Z}^{\oplus[P]} \Big/ \begin{matrix}[P] = \Sigma_i [P_i] \\ [P] = [gP]\end{matrix}$$

$$K_1(\mathcal{P}^{E^1}_{T(1)}) = \mathbb{R} \wedge_{\mathbb{Z}} \mathbb{R}$$

$$x \wedge y$$

$[P \circlearrowleft \text{scissors automorphism}] \in K_1$

$$P \rightleftarrows \sqcup_i P_i$$



$$K_1(\mathcal{P}^{E^1}_{E(1)}) = 0$$

$$K_2(\mathcal{P}^{E^1}_{T(1)}) = ?$$

Higher Hilbert's 3^{rd} problem: Compute <u>higher</u> scissors congruence groups.

Thm (BGMMZ) $K(\mathcal{P}^X_G)$ is a homotopy orbit spectrum

$G \leq \mathrm{Isom}(X)$

1 trivial group

$$\underset{\uparrow\, G\text{-s.c.}}{K(\mathcal{P}^X_G)} \simeq \underset{\uparrow\, \text{no moving s.c.}}{K(\mathcal{P}^X_1)_{hG}}$$

(Farrell-Jones Isomorphism for scissors congruence K-theory!)

Get a trace map

$K_0(\mathcal{P}^X_1) = \mathbb{Z}^{\oplus [P]} / [P] = \Sigma_i [P_i] \xrightarrow{\text{volume}} \mathbb{R}$

$$K(\mathcal{P}^X_1)_{hG} \longrightarrow (H\mathbb{R})_{hG}$$

$$K_m(\mathcal{P}^X_G) \longrightarrow \underset{\uparrow\, \text{group homology}}{H_m(G;\mathbb{R})}$$

(G discrete!)

$$K_0(\mathcal{P}^X_G) \longrightarrow \mathbb{R}$$

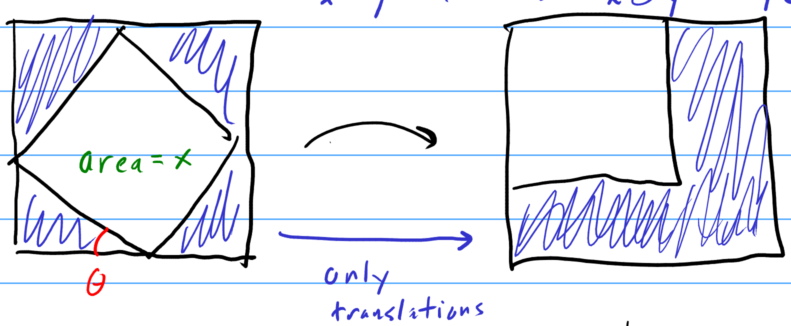
$$[P] \longmapsto \mathrm{vol}(P)$$

$$K_1(\mathcal{P}^X_G) \longrightarrow G^{ab} \otimes_{\mathbb{Z}} \mathbb{R}$$

$$[P = \cup_i P_i \circlearrowleft g_i] \longmapsto \Sigma_i\, g_i \otimes \mathrm{vol}(P_i)$$

$$K_1(\mathcal{P}^{E^1}_{T(1)}) = \mathbb{R} \wedge \mathbb{R} \longrightarrow \mathbb{R} \otimes \mathbb{R}$$

$$x \wedge y \longmapsto x \otimes y - y \otimes x$$



nonzero classes!

$\in K_1(\mathcal{P}^{E^2}_{SE(2)})$

oriented

$$\theta \otimes x \in SE(2)^{ab} \otimes_{\mathbb{Z}} \mathbb{R}$$

$\neq 0$

$$\cong \mathbb{R}/2\pi\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}$$

What about $K(\mathcal{P}^\times_1)$?

$$\pi_0 = \mathcal{P}(X,1) = \mathbb{Z}^{\oplus[P]} / [P] = \Sigma_i [P_i]$$

$$= Pt(X)$$

polytope group

Classically, $\mathcal{P}(X, G) = H_0(G; Pt(X))$.

Fact: $Pt(X)$ is free abelian, $\oplus \mathbb{Z}$.

$\pi_1 = ?$

Thm (MZ) $K(\mathcal{P}^\times_1)$ is a wedge of sphere spectra $\vee \mathbb{S}^0$.

$\Rightarrow \pi_m = Pt(X) \otimes \pi_m^s(S^0)$, $K(\mathcal{P}^\times_G) \simeq (\vee \mathbb{S}^0)_{hG}$

get a spectral sequence

$$H_p(G; Pt(X) \otimes \pi_q^s(S^0)) \Rightarrow K_{p+q}(\mathcal{P}^\times_G)$$

$$H_m(G; Pt(X)) \otimes \mathbb{Q} \cong K_m(\mathcal{P}^\times_G) \otimes \mathbb{Q}$$

group homology! rational s.c.

Cor. $K_m(\mathcal{P}^{E^1}_{T(1)}) \otimes \mathbb{Q} \cong \Lambda^{m+1}_{\mathbb{Z}} \mathbb{R}$

$$K_m(\mathcal{P}^{E^1}_{E(1)}) \otimes \mathbb{Q} \cong \begin{cases} \Lambda^{m+1}_{\mathbb{Z}} \mathbb{R} & \text{if } m \text{ even} \\ 0 & \text{if } m \text{ odd} \end{cases}$$

To get an integral answer, need to make $K(\mathcal{P}_1^X) \simeq \vee S^0$ <u>equivariant</u>.

Topological version of $Pt(X)$:

X geometry

$U \le X$ complete geodesic subspaces

$$PT(X) = \left(\operatorname*{hocolim}_{U \le X} U\right)^{\simeq X} \Big/ \operatorname*{hocolim}_{U \lneq X} U$$

<u>Prop</u>. $PT(X) \simeq \vee S^n \qquad n = \dim X$

$$H_n(PT(X)) \cong Pt(X)$$

Tangent bundle TX pulls back to $PT(X) \setminus \{*\}$

$\rightsquigarrow \Sigma^{-TX} PT(X)$

<u>Thm</u> (M) $K(\mathcal{P}_1^X)$ is <u>equivariantly</u> $\Sigma^{-TX} PT(X)$.

+(BGMMZ) $K(\mathcal{P}_G^X) \simeq \Sigma^{-TX_{hG}} PT(X)_{hG}$

scissors congruence K-theory is a <u>Thom</u> spectrum!

<u>Thm</u>. $K(\mathcal{P}_G^{E^n})$ is <u>rational</u> if $T(n) \le G$.

<u>Cor</u>. $K_m(\mathcal{P}^{E^1}_{T(1)}) \cong \Lambda^{m+1}_{\mathbb{Z}} \mathbb{R}$

$$K_m(\mathcal{P}^{E^1}_{E(1)}) \cong \begin{cases} \Lambda^{m+1}_{\mathbb{Z}} \mathbb{R} & m \text{ even} \\ 0 & m \text{ odd} \end{cases}$$

More:

$K_1 = 0?$

$$K_m(\mathcal{P}^{E^2}_{E(2)}) \cong \frac{H_{m+2}(E(2);\mathbb{Q}^t)}{H_{m+2}(O(2);\mathbb{Q}^t)} \xrightarrow{tr} H_m(E(2);\mathbb{R})$$

$$K_m(\mathcal{P}^{H^2}_{H(2)}) \otimes \mathbb{Q} \cong \frac{H_{m+2}(H(2);\mathbb{Q}^t)}{H_{m+2}(O(2);\mathbb{Q}^t)}$$

$K_1 \otimes \mathbb{Q} = 0$ ✓

$$K_m(\mathcal{P}^{S^2}_{O(3)}) \otimes \mathbb{Q} \cong H_m(O(3);\mathbb{Q}) \oplus H_{m+1}(O(2);\mathbb{Q}^t)$$

Higher Dehn Invariant

$$K(\mathcal{P}^{E^n}) \longrightarrow K(\mathcal{P}^{E^{n-2d}}) \wedge \tilde{K}(\mathcal{P}^{S^{2d-1}})$$