ECHT mini-course

Equivariant Cohomology

9/15/20

Lectures:

1) Intro to $RO(G)$-graded cohomology
2) First calculations
3) Theoretical results

Lecture 1 - Intro to $RO(G)$-graded cohomology

Refs:
- Bredon - Equivariant Cohomology Theories
- May et al. - Equivariant Homotopy and Cohomology Theory (Alaska notes)
- Hill, Hopkins, Ravenel - Kervaire Invariant One paper + book project
- Hill - Handbook of Homotopy Theory Ch. 17

G - finite group

G-Top cat. of spaces w/ a G-action $X \circlearrowleft G$

+ equivariant maps $f: X \to Y$ $\quad f(g \cdot x) = g \cdot f(x)$

$\rightsquigarrow H \leq G \quad f^H: X^H \longrightarrow Y^H$

$f: X \to Y$ is a w.e. if $f^H: X^H \longrightarrow Y^H$ w.e. $\forall H \leq G$

G-CW complexes: attach orbit cells $G/H \times D^n$

$$\begin{tikzcd}
\coprod_{i\in I} G/H_i \times S^{n-1} \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_{i\in I} G/H_i \times D^n \arrow[r] & X_n \arrow[ul, phantom, "\text{p.o.}"]
\end{tikzcd}$$

Equivariant Whitehead Thm: w.e. between G-CW cpxes is a homotopy equivalences

$$\pi_n^H(X) := [S^n, X]^H \cong [S^n, X^H] = \pi_n(X^H)$$

fixed

$$f(g\cdot x) = g\cdot f(x)$$
$$\shortparallel$$
$$f(x) = f(e\cdot x)$$

really pointed maps so use $G\text{-}Top_* \ni X$ – has a G-fixed basepoint

$G \notin G\text{-}Top_* \ni G_+$ disjoint basepoint

Bredon $H^*_G(X;m)$ requires coeff. system

$$H^*_G(G/K \times D^n) \cong H^*_G(G/K) \longrightarrow H^*_G(G/J)$$

$$m : Orb_G^{op} \longrightarrow Ab$$

(finite G-sets, equiv. maps, $\amalg$

To compute: build G-CW complex, apply m to orbit cells + attaching maps $\rightsquigarrow$ C^* take homology

V real orthogonal G representation 

$S^V = \hat{V}$ representation sphere

$$\Sigma^V X = S^V \wedge X = S^V \times X / S^V \vee X$$

$RO(G)$ = Grothendieck group of real orthogonal G reps

$= \mathbb{Z}\langle \text{irreducible reps} \rangle$

<u>Thm</u> (Lewis, May, McClure) Ordinary $\mathbb{Z}$-graded Bredon cohomology extends to an $RO(G)$-graded cohomology theory iff coeffs. M extends to a Mackey functor $\underline{m}$.

$\hookrightarrow$ For $\alpha \in RO(G)$ $\quad \alpha = [V] - [W]$

get $H^\alpha_G(X; \underline{m})$ w/ susp. iso

$$\widetilde{H}^\alpha_G(X; \underline{m}) \cong \widetilde{H}^{\alpha+V}_G(\Sigma^V X; \underline{m})$$

Brown representability: $H^\alpha_G(-; \underline{m})$ represented by genuine equiv. Eilenberg-MacLane spectrum $H\underline{m} \in Sp^G$

If $\underbrace{\alpha = [V]}_{n\text{-dim and trivial}}$ $\qquad H^\alpha_G(-; \underline{m}) \cong H^n_G(-; \underline{m})$

<u>Def</u> A Mackey functor $\underline{m}$ is $\begin{cases} \underline{m}^* : Orb_G^{op} \longrightarrow Ab \\ \underline{m}_* : Orb_G \longrightarrow Ab \end{cases}$

s.t. 1) $\underline{m}^*(T) = \underline{m}_*(T) = \underline{m}(T)$

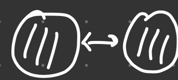
2) $\underline{m}^*$ preserves products $\qquad$ (check $\underline{m}^*$ or $\underline{m}_*$?)

3) "push-pull"

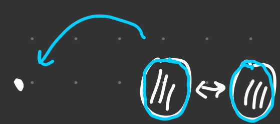
$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\text{p.b.}"] & R \arrow[d, "i"] \\ S \arrow[r, "h"'] & T \end{tikzcd} \implies \begin{tikzcd} \underline{m}(P) \arrow[r, "f_*"] \arrow[dr, phantom, "\circ"] & \underline{m}(R) \\ \underline{m}(S) \arrow[u, "g^*"] \arrow[r, "h_*"'] & \underline{m}(T) \arrow[u, "i^*"'] \end{tikzcd}$$

$G = C_2$

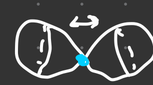
Orbits: $C_2 = C_2/e$ $\qquad$ $pt = C_2/C_2$

$C_2 \times D^n$  $\qquad$ $pt \times D^n$ 

Ex



pt $\qquad$ $C_2 \times D^n$



reps: $\mathbb{R}_{triv}$, $\mathbb{R}_{sgn}$

$$V \cong \mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^{q}_{sgn}$$

$p = $ dim, $q = $ twisted dim/weight, $p-q = $ fixed-set dim/coweight

$\hat{V} = S^V = S^{p,q}$ $\qquad$ $p = $ top'l dim

$S^{1,0}$ $\qquad$ $S^{1,1}$ $\qquad$ $S^{2,0}$ $\qquad$ $S^{2,1}$ $\qquad$ $S^{2,2}$

HW Find C_2-CW structures for $S^{1,1}$, $S^{2,1}$, $S^{2,2}$
(cells + attaching maps)

$\overset{\circlearrowleft t}{C_2} \xrightarrow{p} pt$ $\qquad$ $C_{2+} \wedge S^V \longrightarrow S^V = pt_+ \wedge S^V$

$S^V \longrightarrow C_{2+} \wedge S^V$??

Pontryagin–Thom collapse map

Mackey functors:

$$\begin{array}{ccccc}
pt & \quad & \underline{m}(pt) & \quad & \mathbb{F}_2 \\
\uparrow p & \mathrm{res}^{C_2}_e p = p^* \Big\downarrow \Big\uparrow p_* = \mathrm{tr}^{C_2}_e p & & 1 \Big\downarrow \Big\uparrow 0 & \\
C_2 & & \underline{m}(C_2) & & \mathbb{F}_2 \\
\circlearrowleft & & \circlearrowleft & & \circlearrowleft \\
t & & t^* = t_* = t & & 1 \\
\mathrm{Orb}_{C_2} & \rightsquigarrow & \underline{m} & & \underline{\mathbb{F}_2}
\end{array}$$

"constant"

Exercise $p^* \circ p_* = \mathrm{id} + t$

$H^{\alpha}_{C_2}(X; \underline{m}) = H^{p,q}(X; \underline{m})$ $\qquad H^{p,q}(X; \underline{\mathbb{F}_2}) = H^{p,q}(X)$

Goal: compute $H^{*,*}(X)$

Ex $H^{p,q}(pt) = ??$

$H^{p,0}(X)$ = Bredon cohomology

$$H^{p,0}(pt) = \begin{cases} \mathbb{F}_2 & p = 0 \\ 0 & \text{else} \end{cases}$$

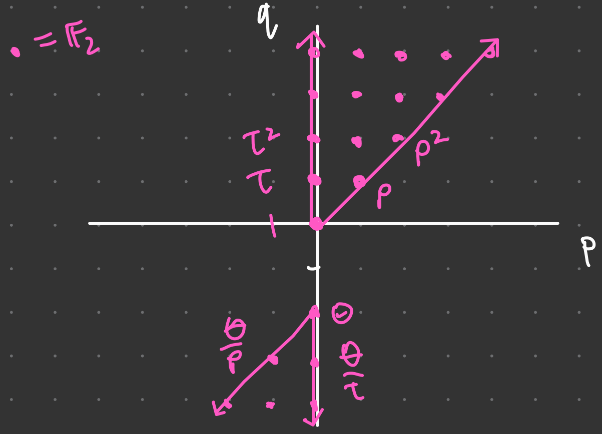
$\underset{pt}{\bullet}$ easy cell structure

$$0 \to \underset{(0)}{\underline{\mathbb{F}_2}(pt) = \mathbb{F}_2} \longrightarrow \underset{(1)}{0} \to \underset{(2)}{0} \to \cdots$$

$$H^{p,0}(C_2) = \begin{cases} \mathbb{F}_2 & p=0 \\ 0 & \text{else} \end{cases}$$

$$0 \to \mathbb{F}_2(C_2) = \mathbb{F}_2 \to 0 \to 0 \to \cdots$$
$$(0) \qquad (1) \qquad (2)$$

$q \neq 0$?? wild stuff



$\mathbb{M}_2 = H^{*,*}(pt; \underline{\mathbb{F}_2})$

infinite-dim.

non-Noetherian ring