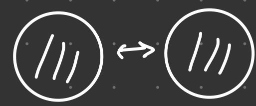
$\boxed{\text{Lecture 2}}$ – First calculations

9/17/20

Last time: $\underline{\underline{G = C_2}}$

$C_2 \times D^n$  $\quad pt \times D^n$ 

$$\begin{array}{ccc}
pt & \underline{m}(pt) & \mathbb{F}_2 \\
p \uparrow & p^* \downarrow \; \uparrow p_* & 1 \downarrow \; \uparrow 0 \\
C_2 & \underline{m}(C_2) & \mathbb{F}_2 \\
\circlearrowleft & \circlearrowleft & \circlearrowleft \\
t & t^* = t_* = t & 1 \\
& & \\
Orb_{C_2} & \underline{m} & \underline{\mathbb{F}}_2
\end{array}$$

$\mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^{q}_{sgn} \rightsquigarrow S^{p,q} = \widehat{\mathbb{R}^{p,q}}$

$S^{1,0} \quad S^{1,1} \quad S^{2,0} \quad S^{2,1} \quad S^{2,2}$

$p = \dim$, $\quad q =$ weight / twisted dim

$RO(C_2)$-graded cohomology $\quad H^{p,q}(X; \underline{\mathbb{F}}_2) = H^{p,q}(X)$

$H^{p,0}(X) =$ Bredon cohom.

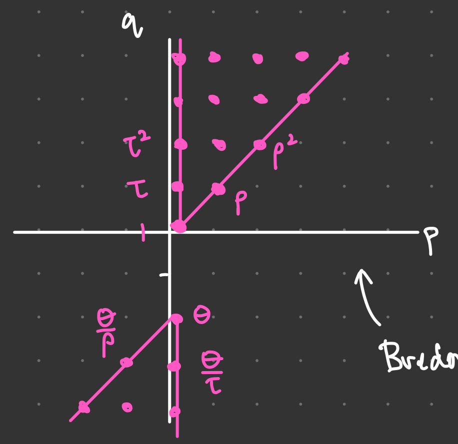
build C_2-CW structure, apply $\underline{\mathbb{F}}_2$, take homology

$$H^{p,0}(pt) = H^{p,0}(C_2) = \begin{cases} \mathbb{F}_2 & p = 0 \\ 0 & \text{else} \end{cases}, \quad \text{but}$$

$\mathbb{M}_2 = H^{*,*}(pt)$ $\quad$ infinite-dim. $\quad$ commutative

non-Noetherian ring

$\bullet = \mathbb{F}_2$



$\mathbb{M}_2 = H^{*,*}(pt)$

$\tau\theta = \rho\theta = 0$

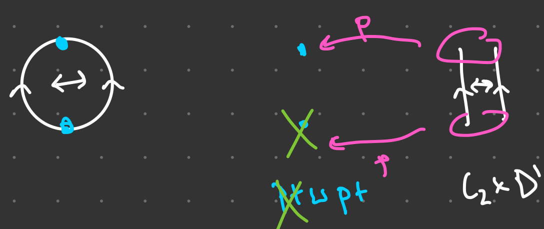
$\theta^2 = 0$

Bredon cohom on p-axis

Due to unpublished work of Stong, here's how to start computing:

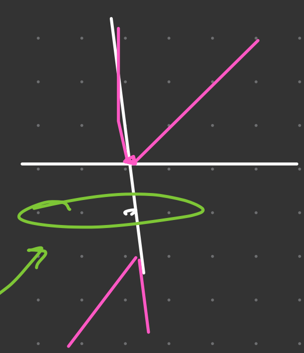
$q=0 \qquad H^{*,0}(pt)$ – Bredon

$q=-1 \qquad H^{*,-1}(pt) \cong \tilde{H}^{*,-1}(S^0) \overset{\text{susp. iso}}{\cong} \tilde{H}^{*+1,0}(S^{1,1})$ – Bredon



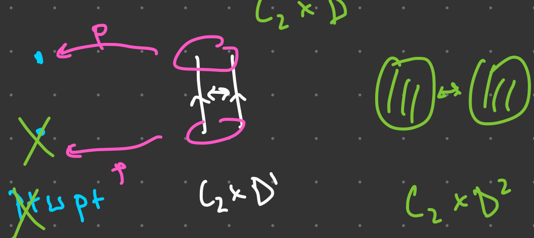
want to work reduced

	(0)		(1)
$\tilde{C}^*(S^{1,1})$	$\underline{m}(pt)$	$\xrightarrow{p^*}$	$\underline{m}(C_2)$
	$\mathbb{F}_2$	$\xrightarrow{1}$	$\mathbb{F}_2$
$\tilde{H}^*(S^{1,1})$	0		0



$q=-2 \qquad H^{*,-2}(pt) \cong \tilde{H}^{*,-2}(S^0) \overset{\text{susp.}}{\cong} \tilde{H}^{*+2,0}(S^{2,2})$ – Bredon

$S^{1,1} \hookrightarrow S^{2,2}$ so build on



$$\begin{array}{llll}
\hat{C}^*(S^{2,2}) & \underline{M}(pt) \xrightarrow{p^*} & \underline{M}(C_2) \xrightarrow{id-t} & \underline{M}(C_2) \\
 & \mathbb{F}_2 \xrightarrow{1} & \mathbb{F}_2 \xrightarrow[1-1]{0} & \mathbb{F}_2 \\
\hat{H}^*(S^{2,2}) & 0 & 0 & \mathbb{F}_2
\end{array}$$

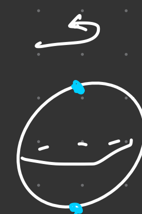
and continue ...

Ref: (In $\underline{\mathbb{Z}}$) Appendix B of arXiv version of Dugger's "An Atiyah-Hirzebruch spectral sequence for KR-theory"

Fact $H^{p,0}(X;\underline{\mathbb{F}_2}) \cong H^p_{sing}(X/C_2;\mathbb{F}_2)$

$S^{1,1}/C_2 \simeq *$

$S^{2,2}/C_2 \simeq S^2$

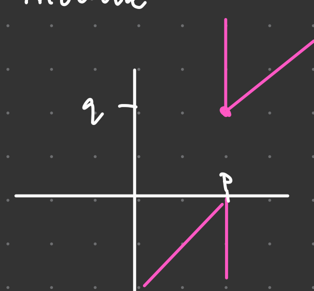


✓

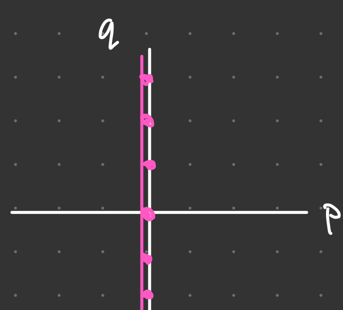
Now let's use $\mathbb{M}_2$:

$X \to pt$ makes $H^{*,*}(X)$ an $\mathbb{M}_2$-module

Susp. iso $\hat{H}^{*,*}(S^{p,q}) \cong \Sigma^{p,q}\mathbb{M}_2$



$H^{*,*}(C_2) \cong \mathbb{F}_2[\tau,\tau^{-1}]$



as an $\mathbb{M}_2$-module

Puppe sequence as usual:

$$\underbrace{X \xrightarrow{f} Y \longrightarrow cof(f)} \longrightarrow \Sigma X \longrightarrow \Sigma Y \longrightarrow \cdots$$

$$\Sigma X = \Sigma^{1,0} X = S^{1,0} \wedge X$$

cofiber sequence $\Rightarrow$ les in reduced cohomology

$$\cdots \leftarrow \tilde{H}^{p+1,q}(\mathrm{cof}) \xleftarrow{d} \tilde{H}^{p,q}(X) \leftarrow \tilde{H}^{p,q}(Y) \leftarrow \tilde{H}^{p,q}(\mathrm{cof}) \leftarrow \cdots$$

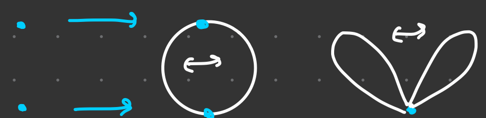
les for each q, but all $\mathbb{M}_2$-module maps

$\rightsquigarrow$ assemble $\quad d : \tilde{H}^{*,*}(X) \longrightarrow \tilde{H}^{*+1,*}(\mathrm{cof})$

get ses $\quad 0 \to \mathrm{cok}(d) \longrightarrow \tilde{H}^{*,*}(Y) \longrightarrow \ker(d) \to 0$

Ex $\quad \tilde{H}^{*,*}(S^{1,1}) \cong \Sigma^{1,1} \mathbb{M}_2$

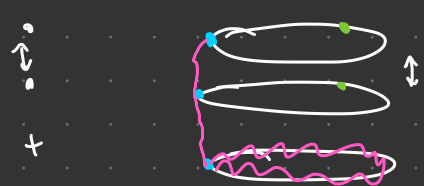
$$S^0 \xrightarrow{\text{"}p\text{"}} S^{1,1} \longrightarrow C_{2+} \wedge S^{1,0} \qquad \text{cof. seq.}$$



$$C_{2+} \wedge S^{1,0} = C_{2+} \times S^{1,0} \big/ C_{2+} \vee S^{1,0} \qquad C_{2+} \vee S^{1,0} = C_{2+} \times pt \cup pt \times S^{1,0}$$

$S^{1,0}$

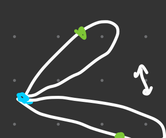
C_{2+}



butterfly

$C_{2+} \wedge S^{1,0}$

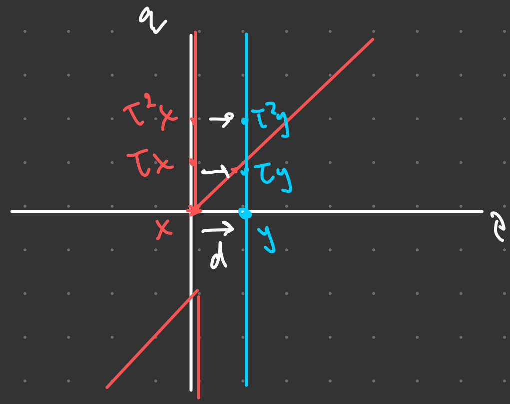
$\simeq$



$C_{2+} \wedge S^{1,1}$

draw les $\quad d : \tilde{H}^{*,*}(S^0) \longrightarrow \tilde{H}^{*+1,*}((C_{2+} \wedge S^{1,0}))$

$$\tilde{H}^{*,*}(S^0) \cong \mathbb{M}_2 \qquad \tilde{H}^{*+1,*}(C_{2+} \wedge S^{1,0}) \cong \Sigma^{1,0} \tilde{H}^{*,*}(C_{2+})$$



$\mathbb{M}_2$ is a free $\mathbb{M}_2$-module

So d is determined

by $d(\text{gen}) = d(x)$

$d(x) = y$??

p-axis is Bredon, $\hat{H}^*_{sing}(S^{1,1}/C_2) = 0$

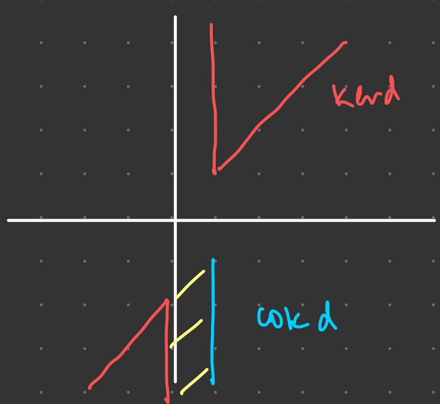
$S^{1,1}/C_2 \cong *$

so yes $d(x) = y$

$d(\tau x) = \tau\, d(x) = \tau y$

$d(\rho x) = \rho\, d(x) = \rho y = 0$

$d(\theta x) = \theta\, d(x) = \theta y = 0$



$$0 \longrightarrow \mathrm{cok}\, d \longrightarrow \hat{H}^{*,*}(S^{1,1}) \longrightarrow \ker d \longrightarrow 0$$

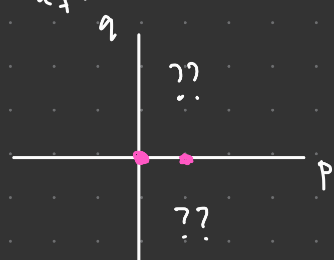
$$\hat{H}^{*,*}(S^{1,1}) \cong \Sigma^{1,1}\mathbb{M}_2$$

extension problems

antipodal action

Ex $H^{*,*}(S^1_a) \cong \hat{H}^{*,*}(S^1_{a_+})$

$S^1_a|_{C_2} \simeq S^1$

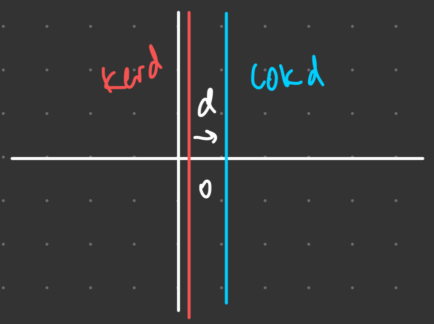


$H^{*,*}(S^1_a)$

cof seq. $C_{2_+} \longrightarrow S^1_{a_+} \longrightarrow C_{2_+} \wedge S^1$

les in $\tilde{H}^{*,*}$

$d: \hat{H}^{*,*}(C_{2_+}) \longrightarrow \hat{H}^{*+1,*}(C_{2_+} \wedge S^1)$



$$0 \longrightarrow \operatorname{cok} d \longrightarrow \tilde{H}^{*,*}(S^1_{a_t}) \longrightarrow \ker d \longrightarrow 0$$

nontrivial extension