$\boxed{\text{Lecture 3}}$ – Further computations / theoretical results

Refs: C. May – A structure theorem for $RO(C_2)$-graded Bredon cohomology

Hazel – The $RO(C_2)$-graded cohomology of C_2-surfaces in $\underline{\mathbb{Z}/2}$-coefficients

Recap: $G = C_2$

$$\mathbb{R}^{p,q} = \mathbb{R}^{p-q}_{triv} \oplus \mathbb{R}^{q}_{sgn} \rightsquigarrow S^{p,q} = \widehat{\mathbb{R}^{p,q}}$$

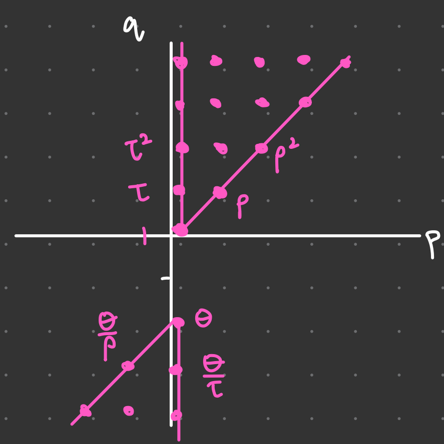
$S^{1,0}$ $\quad$ $S^{1,1}$ $\quad$ $S^{2,0}$ $\quad$ $S^{2,1}$ $\quad$ $S^{2,2}$

$p = \dim$, $\quad q = $ weight / twisted dim

$RO(C_2)$-graded cohomology $\quad H^{p,q}(X;\underline{\mathbb{F}_2}) = H^{p,q}(X)$

$\mathbb{M}_2 = H^{*,*}(pt)$

$\bullet = \mathbb{F}_2$



$\tau\theta = \rho\theta = 0$

$\theta^2 = 0$

$X \longrightarrow Y \longrightarrow Y/X$ $\quad$ cofiber seq.

les $\quad \cdots \leftarrow \widetilde{H}^{*+1,*}(Y/X) \xleftarrow{d} \widetilde{H}^{*,*}(X) \longleftarrow \widetilde{H}^{*,*}(Y) \longleftarrow \widetilde{H}^{*,*}(Y/X) \longleftarrow \cdots$

ses $\quad 0 \to \operatorname{cok} d \to \tilde{H}^{*,*}(Y) \to \ker d \to 0$

often extension problems

<u>Fact</u> $\quad H^{p,0}(X;\mathbb{F}_2) \cong H^p_{sing}(X/C_2;\mathbb{F}_2)$

<u>Fact</u> (ρ-les) (Araki-Murayama)

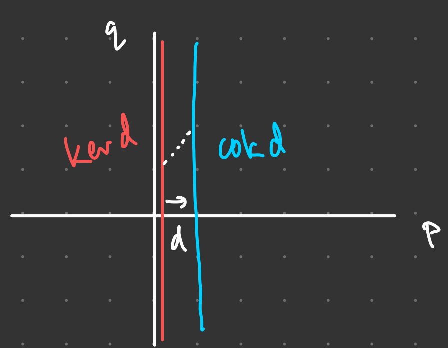
$$\cdots \to \hat{H}^{*,*}(X) \xrightarrow{\cdot\rho} \hat{H}^{*+1,*+1}(X) \xrightarrow{\psi} \hat{H}^{*+1}_{sing}(X) \to \hat{H}^{*+1,*}(X) \to \cdots$$

$$\rho \mapsto 0$$
$$\tau \mapsto 1$$
$$\theta \mapsto 0$$

<u>Last time</u>: $\quad H^{*,*}(S^1_a) \cong \hat{H}^{*,*}(S^1_{a_+})$

$$C_{2_+} \longrightarrow S^1_{a_+} \longrightarrow C_{2_+} \wedge S^1 \quad \text{cof. seq.}$$

les $\quad d : \hat{H}^{*,*}(C_{2_+}) \longrightarrow \hat{H}^{*+1,*}(C_{2_+} \wedge S^1)$



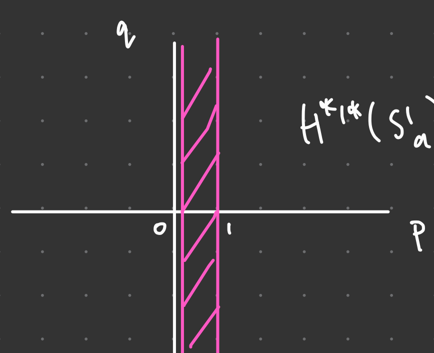
$S^1_a / C_2 \simeq S^1$

So $\quad d = 0$

$$0 \to \operatorname{cok} d \to \tilde{H}^{*,*}(S^1_{a_+}) \to \ker d \to 0$$

<u>Exercise</u> $\quad \rho$ les $\Rightarrow$ ρ-connection

$\rho\tau = \tau\rho \quad \Rightarrow$ all ρ-connections

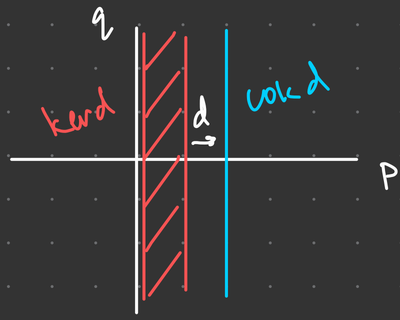


$H^{*,*}(S^1_a) =: \mathbb{A}_1$

Ex $H^{*,*}(S^2_a)$ similar

$$S^1_{a+} \longrightarrow S^2_{a+} \longrightarrow C_{2+} \wedge S^2$$

les

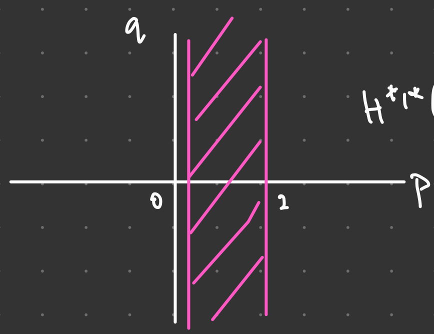


$S^2_a / C_2 \simeq \mathbb{RP}^2$

so $d = 0$

ρ les $\Rightarrow$ ρ connection

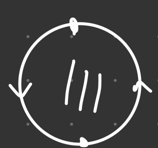
$\rho\tau = \tau\rho \Rightarrow$ all ρ connections



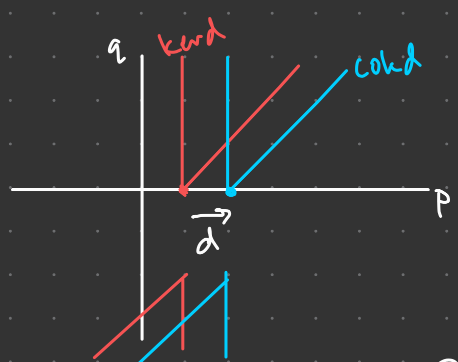
$H^{*,*}(S^2_a) =: \mathbb{A}_2$

Exercise Induct to get $H^{*,*}(S^n_a) =: \mathbb{A}_n$

Ex $\mathbb{RP}^2_{triv}$



$$S^{1,0} \longrightarrow \mathbb{RP}^2_{triv} \longrightarrow S^{2,0}$$



d is determined by $d(\text{gen})$

$\mathbb{RP}^2_{triv} / C_2 \simeq \mathbb{RP}^2$

so $d = 0$

$$0 \to \operatorname{cok} d \to \tilde{H}^{*,*}(\mathbb{RP}^2_{triv}) \to \ker d \to 0$$

$$0 \to \Sigma^{2,0}\mathbb{M}_2 \to \tilde{H}^{*,*}(\mathbb{RP}^2_{triv}) \to \Sigma^{1,0}\mathbb{M}_2 \to 0$$

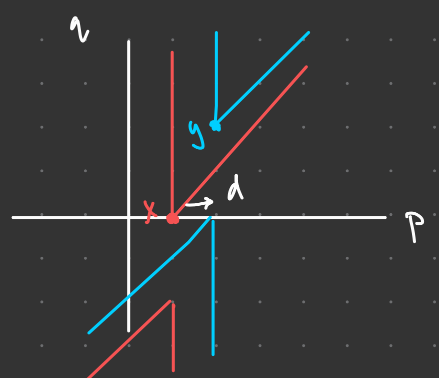
free

So $\tilde{H}^{*,*}(\mathbb{RP}^2_{triv}) \cong \Sigma^{1,0}\mathbb{M}_2 \oplus \Sigma^{2,0}\mathbb{M}_2$

In fact, $\hat{H}^{*,*}(X_{triv}) \cong \hat{H}^*_{sing}(X) \otimes \mathbb{M}_2$

Ex $\mathbb{RP}^2_{tw}$

$$S^{1,0} \longrightarrow \mathbb{RP}^2_{tw} \longrightarrow S^{2,2}$$



d determined by $d(gen) = d(x)$

Is $d(x) = \theta y$?

$\mathbb{RP}^2_{tw} / C_2 \simeq *$

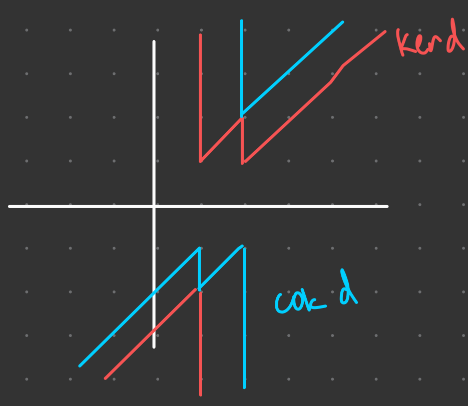
So yes $d(x) = \theta y$

$d(\tau x) = \tau\theta y = 0$

$d(\rho x) = \rho\theta y = 0$

$d(\theta x) = \theta\cdot\theta y = 0$

$\dots$

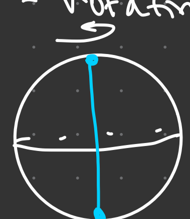


$$0 \to \operatorname{cok} d \to \hat{H}^{*,*}(\mathbb{RP}^2_{tw}) \to \ker d \to 0$$

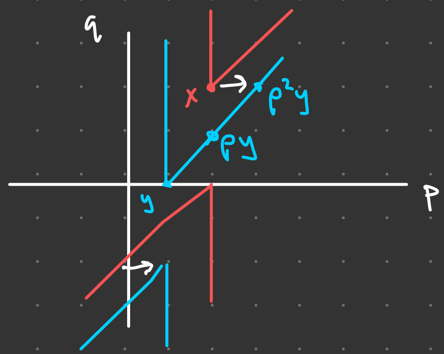
nontrivial extension

Exercise Find another cof. seq. and compare

Ex HDrot = rotating pinched sphere "Hole-less donut"



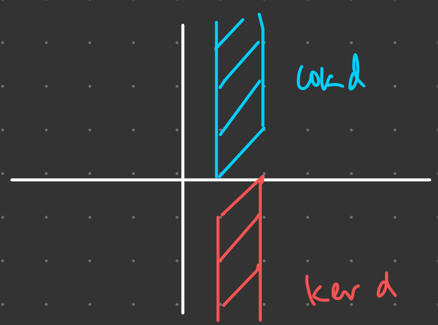
$$S^{0,0} \xrightarrow{\text{"}\rho^2\text{"}} S^{2,2} \longrightarrow HD_{rot} \longrightarrow S^{1,0}$$



$d(x) = \rho^2 y$ "by construction" or

compare w/ another cofiber seq.

$$d\left(\frac{\theta}{\rho^2}x\right) = \frac{\theta}{\rho^2}\cdot\rho^2 y = \theta y$$



$$0 \to \operatorname{cok} d \to \tilde{H}^{*,*}(HD_{rot}) \to \ker d \to 0$$

nontrivial extensions

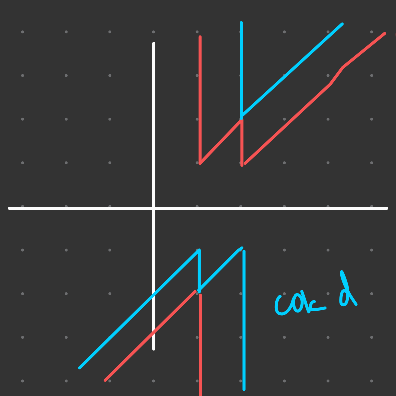
τ-connections

Lots of $\mathbb{M}_2$-modules, lots of nontrivial extensions

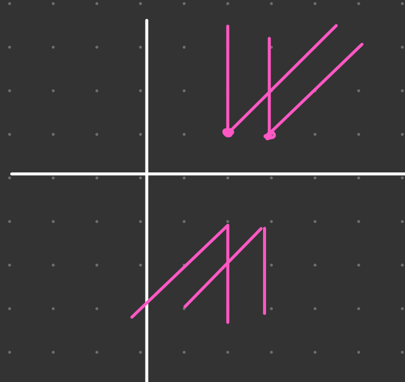
<u>Thm</u> (m.)

As an $\mathbb{M}_2$-module, $H^{*,*}(X)$ decomposes as a direct sum of shifted copies of $\mathbb{M}_2 = H^{*,*}(pt)$ and $\mathbb{A}_n$

(X: finite C_2-CW cpx)



$\rightsquigarrow$



$$\tilde{H}^{*,*}(\mathbb{RP}^2_{tw}) \cong \Sigma^{1,1}\mathbb{M}_2 \oplus \Sigma^{2,1}\mathbb{M}_2$$

$\tilde{H}^{*,*}(HD_{rot}) \cong \Sigma^{1,0}\mathbb{A},$

<u>Prop</u> $\mathbb{M}_2$ is self-injective

So $\quad 0 \longrightarrow \Sigma^{p,q}\mathbb{M}_2 \longrightarrow \hat{H}^{*,*}(X) \longrightarrow Q \longrightarrow 0$