UNIVERSAL EXPONENTIAL FORMULAS:

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What is an exponential formula?

power sum $p_k(x_1, \ldots, x_n) = x_1^k + x_2^k + \ldots + x_n^k$

homogeneous symmetric polynomial $h_k(x_1, \ldots, x_n) = \prod_{i_1 + \ldots + i_n = k} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n}$

sum of homogeneous monomials of degree k

$h_3(x_1, x_2) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3$

$h_1(x_1, \ldots, x_n) = p_1(x_1, \ldots, x_n)$

$h_k = \frac{1}{k} \sum_{i=1}^{k} h_{k-i} p_i$

Exponential Formula:

$$\sum_{m \geq 0} h_m t^m = \exp\left(\sum_{k \geq 1} p_k \frac{t^k}{k} \right)$$

$\exp(t) = \sum_{m \geq 0} \frac{t^m}{m!}$

There is a robust analogy: $h_m \leftrightsquigarrow$ symmetric powers

$p_k \leftrightsquigarrow$ Adams operations ψ^k

<u>Recall:</u> G a finite group, $RU(G)$ representation ring, $k \geq 0$

The Adams operations $\psi^k : RU(G) \longrightarrow RU(G)$ are characterized by

- ψ^k is a ring homomorphism $\Longrightarrow \psi^k(1) = 1 \Longrightarrow \psi^k(n) = n$
- if $\dim(V) = 1$, $\psi^k(V) = V^{\otimes k}$
- ψ^k is natural in G

Analogy $\Longrightarrow$ $$\sum_{m \geq 0} \mathrm{Sym}^m t^m = \exp\left(\sum_{k \geq 1} \psi^k \frac{t^k}{k}\right)$$ gives ψ^k in terms of Sym^m

V a trivial G-rep $\dim V = n$

$$\sum_{m \geq 0} \binom{m+n-1}{m} t^m = \exp\left(\sum_{k \geq 1} n \frac{t^k}{k}\right)$$

$\dim \mathrm{Sym}^m(V)$
$= h_m(1, 1, \ldots, 1)$

$n = \dim \psi^k(V)$
$n = p_k(1, 1, \ldots, 1)$

Interpret $RU(G)$ as $KU_G(*)$; formulas upgrade to statements about cohomology operations on KU_G.

This is the height one integral version of the following:

<u>Theorem</u> (Ganter): In Morava E-theory at height n, prime p

$$\sum_{m \geq 0} \underbrace{\beta_m}_{\text{"symmetric powers"}} t^m = \exp\Big(\sum_{k \geq 0} \underbrace{T_{p^k}}_{\text{Hecke operations}} \frac{t^{p^k}}{p^k} \Big)$$

<u>Goal</u>: new, categorical proof of this via universal properties

<u>Question</u>: Where do these formulas live?

Work with the representation ring case.

Note that Sym^m factors through the (total) power operations

$$\begin{array}{ccccc}
\mathrm{Sym}^m : RU(G) & \xrightarrow{P_m} & RU(G\times\Sigma_m) & \xrightarrow[\mathrm{Ind}^{G}_{G\times\Sigma_m}]{-\otimes_{\mathbb{C}[G\times\Sigma_m]}\mathbb{C}[G]} & RU(G) \\
V & \longmapsto & V^{\otimes m} \curvearrowleft \Sigma_m & \longmapsto & \mathrm{Sym}^m(V)
\end{array}$$

<u>Remark:</u> P_m is only a multiplicative homomorphism

$$RU(G) \xrightarrow{\underbrace{\sum_{m\geq 0} P_m t^m}} \underbrace{\prod_{m\geq 0} RU(G\times\Sigma_m)t^m} \xrightarrow{\sum \mathrm{Ind}^{G}_{G\times\Sigma_m} t^m} RU(G)[\![t]\!]$$

"Summa power operation" sends addition to multiplication

commutative ring via

$$Vt^i \cdot Wt^j = \mathrm{Ind}^{G\times\Sigma_{i+j}}_{G\times\Sigma_i\times\Sigma_j}(V \boxtimes W)t^{i+j}$$

"transfer multiplication"

ring homomorphism

map down to get $\sum_{m\geq 0} \mathrm{Sym}^m t^m$

G acts diagonally
Σ_i acts on left
Σ_j acts on right

<u>Remark:</u> the name "total power operation" is already taken. Summa is Latin for total, and also this is a "sum of power operations" (credit to Adrielle Stapleton for the name)

Adams ops factor through power operations in a different way

$$\psi^k : RU(G) \xrightarrow{P_k} RU(G \times \Sigma_m) \longrightarrow RU(G \times \Sigma_m)/I_{tr} \cong RU(G)$$

smallest ideal such that the composite is a ring homomorphism, generated by images of $\mathrm{Ind}_{\Sigma_i \times \Sigma_j}^{\Sigma_m}$ $\quad i+j=m$, $i,j>0$

P_m is multiplicative homomorphism,

$\psi^k = P_k/I_{tr}$ is ring homomorphism

"additive power operation"

Note $\exp(t) = \sum_{m \geq 0} \frac{t^m}{m!} \notin \mathbb{Z}[[t]]$. It lives in $\mathbb{Z}\langle\langle t \rangle\rangle$

ring of divided power series $\sum_{m \geq 0} a_m \frac{t^m}{m!}$.

So $\exp\left(\sum_{k \geq 1} \psi^k \frac{t^k}{k}\right)$ a priori lives in $RU(G)\langle\langle t \rangle\rangle$.

What about an exponential formula for $\sum_{m \geq 0} P_m t^m$? Where should this live?

Integer-valued class functions on Σ_m

$$\prod_{m \geq 0} RU(G \times \Sigma_m) t^m \xrightarrow{\cong} \prod_{m \geq 0} RU(G) \otimes RU(\Sigma_m) t^m \xhookrightarrow{1 \otimes \chi} \prod_{m \geq 0} RU(G) \otimes Cl(\Sigma_m, \mathbb{Z}) t^m$$

Product on $\prod_{m\geq 0} RU(G)\otimes Cl(\Sigma_m,\mathbb{Z})\,t^m$

assume G is trivial for convenience

Recall $\{\text{conjugacy classes in } \Sigma_m\} \cong \{\text{partitions } \lambda\vdash m\}$

$\lambda\vdash m \quad \lambda=(\lambda_1,\lambda_2,\lambda_3,\_\_\_)$ with $\sum \lambda_i i=m$

$\lambda = 1+1+1+3+3 \vdash 9$

$\lambda_1=3,\ \lambda_3=2$

$\lambda! = 3!\,2!$

$Cl(\Sigma_m,\mathbb{Z})$ has a basis of characteristic functions $\{\mathbb{1}_\lambda\}_{\lambda\vdash m}$

vector addition

transfer product on $\prod_{m\geq 0} Cl(\Sigma_m,\mathbb{Z})t^m$ given by $\mathbb{1}_\lambda t^m\cdot \mathbb{1}_\tau t^n = \dfrac{(\lambda+\tau)!}{\lambda!\,\tau!}\mathbb{1}_{\lambda+\tau}t^{m+n}$

$\lambda! = \prod_i \lambda_i!$

$\tau = 1+2\vdash 3$ $\quad (\lambda+\tau)_1 = 4$

$\tau_1=1,\ \tau_2=1$

$(\lambda+\tau)! = 4!\,1!\,2!$

<u>Claim:</u> $\prod_{m\geq 0} Cl(\Sigma_m, \mathbb{Z})t^m$ is a <u>divided power algebra</u>

graded algebra such that if $|x|>0$, $k! \mid x^k$

vector addition

$$\mathbb{1}_\lambda t^m \cdot \mathbb{1}_\tau t^n = \frac{(\lambda+\tau)!}{\lambda!\,\tau!}\mathbb{1}_{\lambda+\tau}t^{m+n}$$

$\lambda! = \prod_i \lambda_i!$

scalar multiplication

$$\mathbb{1}_\lambda^k t^m = \frac{(k\cdot\lambda)!}{(\lambda!)^k}\mathbb{1}_{k\cdot\lambda}t^{km}$$

$m>0$

order of $\Sigma_{k\lambda}$

order of $\Sigma_\lambda^{\times k}$

$$\Sigma_{k\lambda} \supseteq \Sigma_\lambda \wr \Sigma_k \supseteq \Sigma_\lambda^{\times k}$$

index $k!$

$\Sigma_\lambda = \prod_i \Sigma_{\lambda_i}$

Proposition: As functions $RU(G) \longrightarrow \prod_{m\geq 0} RU(G)\otimes Cl(\Sigma_m, \mathbb{Z})\, t^m$,

these functions send addition to multiplication

$$\sum_{m\geq 0} P_m t^m = \exp\left(\sum_{k\geq 1} \psi^k \otimes \mathbb{1}_{k\vdash k}\, t^m\right)$$

partition $\lambda \vdash k$, $\lambda_k = 1$

Can recover previous exponential formula:

$$\begin{tikzcd}
RU(G) \arrow[r] & \prod_{m\geq 0} RU(G\times\Sigma_m)\, t^m \arrow[r] \arrow[d] & RU(G)[\![t]\!] \arrow[d] \\
& \prod_{m\geq 0} RU(G)\otimes Cl(\Sigma_m, \mathbb{Z})\, t^m \arrow[r, dashed] & RU(G)(\!(t)\!)
\end{tikzcd}$$

exponential formula here

yields exponential formula here

Question: How general is this? When do we have exponential formulas?

Replace RU by a global Green functor R:

$R = (R^*, R_*)$ $\quad R^* : \mathrm{Group}^{op} \longrightarrow \mathrm{CRing}$, $\quad R_* : \mathrm{Group}^{\hookrightarrow} \longrightarrow \mathrm{Ab}$

$R^*(G) = R_*(G)$

$tr_f = R_*(H \xhookrightarrow{f} K) : R(H) \longrightarrow R(K)$

$+$ conditions

Examples: RU, $Cl(-,\mathbb{Z})$, $Cl(-,\mathbb{C})$, A, $H^*(-;k)$, $E^*(B(-))$, ...

Burnside ring

field

homotopy commutative ring spectrum

If R is a global Green functor, build a commutative ring $R[\![\Sigma]\!] = \prod_{m \geq 0} R(\Sigma_m) t^m$

product $\quad at^i \cdot bt^j = \mathrm{tr}_{\Sigma_i \times \Sigma_j}^{\Sigma_{i+j}} (a \boxtimes b) t^{i+j}$

$$R(\Sigma_i) \otimes R(\Sigma_j) \xrightarrow{pr_L^* \otimes pr_R^*} R(\Sigma_i \times \Sigma_j)$$

$$a \otimes b \longmapsto a \boxtimes b$$

What plays the role of class functions?

$$R\langle\!\langle \Sigma \rangle\!\rangle = \prod_{m \geq 0} \bigoplus_{\lambda \vdash m} \left(R(\Sigma^\lambda) / I_{tr} \right)^{\Sigma_\lambda} t^m$$

$\Sigma^\lambda = \prod_i \Sigma_i^{\times \lambda_i}$

$\ni \mathbb{1}_\lambda$ unit of this ring

When $R = RU$, $\left(RU(\Sigma^\lambda) / I_{tr} \right)^{\Sigma_\lambda} \cong \mathbb{Z}$

$$\bigoplus_{\lambda \vdash m} \mathbb{Z} \cong Cl(\Sigma_m, \mathbb{Z})$$

Theorem: $R\langle\langle \Sigma \rangle\rangle$ admits a canonical ring structure with a ring homomorphism $R[[\Sigma]] \longrightarrow R\langle\langle \Sigma \rangle\rangle$. Furthermore, $R\langle\langle \Sigma \rangle\rangle$ is a divided power algebra.

additive power operation (ring hom)

If R is a global power functor, can build $P_m/I_{tr} : R(G) \longrightarrow R(G\times\Sigma_m) \longrightarrow R(G\times\Sigma_m)/I_{tr}$

global Green functor + power operations $P_m : R(G) \longrightarrow R(G\times\Sigma_m)$

Corollary: If R is a global power functor such that $R(G\times\Sigma_m) \cong R(G)\otimes R(\Sigma_m)$ for all G, m then

$$\sum_{m\geq 0} P_m t^m = \exp\Big(\sum_{k\geq 1} P_k/I_{tr} \otimes \mathbb{1}_{k\vdash k}\, t^k\Big)$$

partition $\lambda \vdash k$

$\lambda_k = 1$

as functions $R(G) \longrightarrow R(G) \hat{\otimes}_{R(e)} R\langle\langle \Sigma \rangle\rangle$

be careful with completions

<u>Universal property</u>: For an ordinary commutative ring S with a homomorphism*

$$R(G) \mathbin{\hat{\otimes}_{R(e)}} R[\![\Sigma]\!] \longrightarrow S[\![t]\!],$$

$$\begin{tikzcd}
R(G) \arrow[r] & R(G) \mathbin{\hat{\otimes}_{R(e)}} R[\![\Sigma]\!] \arrow[r] \arrow[d] & S[\![t]\!] \arrow[d] \\
 & R(G) \mathbin{\hat{\otimes}_{R(e)}} R\langle\!\langle \Sigma \rangle\!\rangle \arrow[r, dashed, "\exists !"] & S\langle\!\langle t \rangle\!\rangle
\end{tikzcd}$$

homomorphism of divided power algebras

exponential formula here - - - - - - - - - yields exponential formula here

<u>Corollary</u>: Recover Ganter's formula for $R = E^*(B(-))$, $S = E^*$

*of bialgebras!