Open and In Progress Research Questions
eCHT Research Workshop on Hopf Rings
June 24 - 28, 2024

Contributors: Tilman Bauer, Jack Carlisle, Paul Goerss, Weinan Lin, Peter May, Jeremy Miller, Peter Patzt, Sarah Petersen, Dev Sinha, and W. Stephen Wilson

Introduction

This compilation of open and in progress research questions on the topics of Hopf algebras, rings, and algebroids appearing in algebraic topology is meant for discussion during the eCHT Research Workshop on Hopf Rings running from 11 AM - 2 PM Eastern Time on June 24 - 28, 2024. A wide range of problems in terms of difficulty, specificity, and topic are described within this document. It is encouraged to skim the entire document and focus on the questions that most interest you. Problems are generally grouped by topic:

Research Questions:

1. Ravenel–Wilson Hopf Ring Techniques

(*Contributed by Jack Carlisle, Sarah Petersen, and W. Stephen Wilson*)

I. **Overarching goal**: For each ring spectrum, compute the Hopf ring of the homology of the (non negatively-graded) spaces representing its omega spectrum (see [36, 29, 30, 21, 22, 37, 38, 27] for examples).
   A. Go further: Understand the extent to which Hopf ring structures exist in the homology of the (non negatively-graded)spaces representing the omega spectra of ring spectra in additional categories such as C_2-equivariant and motivic spectra.
      1. The story in C_2-equivariant and motivic spectra is more complicated as one generally does not have a Künneth isomorphism.
      2. Some important examples, such as the $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg–MacLane spaces [27], and the motivic cohomology of motivic Eilenberg–MacLane spaces as well as the Hopf ring of algebraic cobordism [1] enjoy the structure of a Hopf ring.
      3. Some outstanding questions related to the $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg–MacLane spaces include:
         a. The mod p-homology of classical nonequivariant Eilenberg–MacLane spaces has a global description. Specifically, Ravenel and Wilson show

Theorem 1 (Ravenel-Wilson [36])**.** *H_*K_* is the free Hopf ring on $H_*K_0 = H_*[\underline{\mathbb{F}}_p]$, H_*K_1, and $H_*\mathbb{C}P^\infty \subset H_*K_2$ subject to the relation that $e_1 \circ e_1 = \beta_1$.*

Can a similar statement be obtained in the C_2-equivariant case? If so, what specifically, is the global structure of the Hopf rings that do arise? One may also ask how the Hopf rings here relate to Hill and Hopkins' work extending Ravenel and Wilson's construction of a universal Hopf ring over MU^* to C_2-equivariant homotopy theory [16].

b. How does the stable relation

$$\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1}$$

in the C_2-equivariant dual Steenrod algebra

$$\mathcal{A}_\star^{C_2} = H\underline{\mathbb{F}}_2[\tau_0, \tau_1, \cdots, \xi_1, \xi_2, \cdots]/(\tau_i^2 = (u + a\tau_0)\xi_{i+1} + a\tau_{i+1})$$

arise from the unstable calculation of $H_\star K_V$?

c. Is it possible to explicitely compute the differentials in the twisted bar spectral sequences computing $H_\star K_{*\sigma}$, perhaps in terms of some norm-like structure?

B. Where all can the Hopf ring structure in the bar spectral sequence be applied?
   1. The Hopf ring structure in the bar spectral sequence significantly restricts potential differentials. For example, differentials must hit primitives in filtrations 0, 1, or 2 (see [36, Sections 7 and 8], [35], and [38, Section 3] for an introduction).
   2. This Hopf ring structure in the bar spectral sequence has been used to compute the mod p-homology of nonequivariant Eilenberg–MacLane spaces [36, Section 8], the Morava K-theory of Eilenberg–MacLane spaces [30], and the $H\underline{\mathbb{F}}_2$-homology of C_2-equivariant Eilenberg–MacLane spaces [27].

C. Can we compute $K(n)_*QS^0$, the Morava K-theory of the sphere spectrum (regarded as a ring spectrum)? Better yet, can we compute the Hopf ring of the Morava K-theory of all the spaces representing the omega spectrum QS^0?

D. Can we compute $K(n)_*ER(2)_*$, the Morava K-theory of the spaces representing the Johnson–Wilson theory $ER(2)$? (We know the mod 2-homology of these spaces [22].) Note the Morava K-theory of $ER(1)$ is known since its simply BO [21].

E. Find the names for algebra generators given in [21, 37, 38], for instance. Motivation: names better than "x" are useful in applications.

F. Starting from the description of the $H\underline{\mathbb{F}}_2$-homology of the zeroth space of $MU_\mathbb{R}$, Real cobordism, given right after the statement of [22, Theorem 1.5], can we apply the $RO(C_2)$-graded bar and twisted bar spectral sequences (see [27, Section 6])) to compute more all or some of $H\underline{\mathbb{F}}_{2\star}^{C_2} MU_{\mathbb{R}\star}$? Do these computations carry Hopf structures?

II. **Geometric cobordism and equivariant cobordism of maps**. This is an ongoing research project led by Jack Carlisle and Sarah Petersen. We are looking for up to three (junior) researchers to join our project. Everyone is welcome to think about it during the workshop. If you are interested in collaborating after the workshop, please email Jack Carlisle and Sarah Petersen at `jcarlisl@nd.edu` and `sarahllpetersen@gmail.com` to express interest.

A. **Project description**: Let A be an abelian group. The cobordism ring of A-manifolds (unoriented, without boundary) is represented by the A-spectrum mO_A, whose Vth space (for V an orthogonal A-representation) is

$$mO_A(V) = \mathrm{Thom}\left(\gamma \to \mathrm{Gr}_{\dim V}(V \oplus \mathbb{R}^\infty)\right).$$

This project aims to prove the cobordism group of A-equivariant maps between A-manifolds is the mO_A-homology of a space representing mO_A, similar to the case for complex cobordism, where the cobordism group of complex maps of codimension $2n$ is the complex bordism group MU_*MU_n, and MU_*MU_* is the cobordism group of all maps with even codimension (see [29, discussion between Theorem 2.1 and Proposition 2.2]).

B. This project is closely related to work of Stong [34, 33], Rinne [31], and Hara [15, 14].

2. Hopf Rings and Dieudonné Theory

(*Contributed by Tilman Bauer, Paul Goerss, and Sarah Petersen*)

III. **Dieudonné theory of dg Hopf algebras** (*Contributed by Tilman Bauer*)

Dieudonné theory turns abelian Hopf algebras over fields into modules over some ring R, so it is a natural question whether the bar- or Eilenberg-Moore spectral sequences used to compute Hopf rings can be more simply expressed on the level of Dieudonné modules. The problem is that the differentials in these spectral sequences are not Hopf algebra homomorphisms, but rather derivations, so they cannot be described as homomorphisms between their Dieudonné modules. However, the category of abelian dg Hopf algebras should still be a category of modules over some ring R^δ.

(a) Compute R^δ. This probably involves finding projective generators of the category of abelian dg Hopf algebras. What are the two functors $\mathrm{Mod}_{R^\delta} \to \mathrm{Mod}_R$ given by (1) forgetting the differential and (2) taking homology?

(b) Compute th R^δ-module associated to the bar (or cobar) construction of the homology of a loop space (in particular, $E_*\underline{E}_m$), i.e. the E^2-term of the bar and Eilenberg-Moore spectral sequences.

My PhD student Elmo Vuorenmaa has thought about the first problem but has put the problem aside for the time being. If you are interested, you should talk to him.

IV. **Dieudonné theory for plethories** (*Contributed by Tilman Bauer*)

There is a (semilinear) composition map

$$E^m(\underline{E}_k) \times E^k(\underline{E}_l) \to E^m(\underline{E}_l)$$

for any spectrum E, and this structure is not part of the (dual of the) Hopf ring structure. I have written "$\times$" here on the left hand side, but of course there really is a composition pairing "$\circ$" of Hopf rings so that the composition map is a map of Hopf algebras. Roughly, Hopf rings with this additional structure are called plethories. See [2], although the setup in that paper deals with plethories over more general rings than fields, which makes the treatment unnecessarily complicated for our case. The plethory structure is crucial input for the unstable E-based Adams spectral sequence, whose E_2-term can be described as an Ext-group of modules over the plethory $E^*\underline{E}_*$. Since Dieudonné theory requires a base field, the most relevant interesting example here is $E = K(n)$, Morava K-theory, or its two-periodic relative K_n.

(a) Describe the composition pairing on the level of Dieudonné modules. I only know how to do this for connective Hopf algebras over a perfect field k, but definitely not for $K(n)^*(\underline{K(n)})$.

(b) My former student Magnus Carlson has shown [7] that all plethories in characteristic 0 are linear, i.e. of the form $\mathrm{Sym}(H)$ for some (noncommutative) bialgebra H. This is false in characteristic p. Find suitable, hopefully weak, conditions on plethories allowing for a full characterization.

(c) Describe the composition map on the known Hopf ring for $K(n)^*(\underline{K(n)})$, with or without Dieudonné module techniques. We know what it has to be after stabilization to the Hopf algebroid $K(n)_*K(n)$, but the stabilization map has a kernel.

V. **Diudonné Theory, Brown–Gitler spectra, and computations** (*given by Mike Hopkins and Paul Goerss, contributed by Paul Goerss*)

For simplicity, let $p = 2$, although there is similar story at all primes. Let $B(n)$ be the classical Brown-Gitler spectra and $T(2n) = \Sigma^{2n}DB(n)$ be the suspension of the Spanier-Whitehead dual. Then the result of [10] is that

$$[T(2n), \Sigma^k E] \cong D_{2n}H_*(\Omega^\infty\Sigma^k E)$$

where $D_*(-)$ is the graded Dieudonné module functor and $H_*(-)$ is mod 2 homology. We then have

$$[T(2n), \Sigma^k E] \cong [S^{2n-k}, E \wedge B(n)] = E_{2n-k}B(n).$$

Thus, if E is a ring spectrum, the even part of the module $D_*H_*\Omega^\infty\Sigma^*E)$ for the Hopf ring for E can be recovered from $E_*B(*)$. This bi-graded Dieudonné module is an actual ring object in the symmetric monoidal structure on graded Dieudonné modules. This was one point of [12]. In terms of Brown-Gitler spectra this is induced by the pairing

$$B(m) \wedge B(n) \to B(m + n).$$

This leads to the obvious problem: find E for which $E_*B(n)$ can be calculated. Then you know the Hopf ring, or a good deal about it. We might concentrate on $k \geq 0$ to start, the range where we know a good deal about the homotopy of $B(n)$. Since the $B(n)$ are wedge summands of the 2-completion of $\Sigma^\infty_+\Omega^2S^3$, computing $E_*\Omega^2S^3$ is a very closely related problem.

For example, if $E = tmf$ then $H^*tmf \cong A//A(2)$ where $A(2)$ is the subalgebra of the Steenrod algebra generated by Sq^{2^i}, $0 \leq i \leq 2$. We then have an Adams-Spectral Sequence

$$\mathrm{Ext}^s_{A(2)_*}(\Sigma^t\mathbb{F}_2, H_*B(n)) \Longrightarrow tmf_{t-s}B(n).$$

Can this be computed? Can you (dis-)prove the conjecture Steve Wilson made in 6.3? The source of his map would be known.

Needless to say, this has been tried before, but we know a great deal more about computing such Adams Spectral Sequences now; see [6] below for example. Similar questions exist for ko, ku, $BP\langle m\rangle$ and so on. This was what Ravenel did for BP. The case ko is discussed in [21]. He found the Mahowald cofiber sequences

$$B(n-1) \to B(n) \to \Sigma^n B([\frac{n}{2}])$$

to be useful for induction arguments.

VI. **Equivariant algebra** (*Contributed by Sarah Petersen*)

A. Classically, the Witt vectors (or rather a dual notion) play a fundamental role in the Dieudonné theory associating a category of modules over some ring to category of graded, connected, bicommutative Hopf algebras over $\mathbb{F}_p$. [12]. In equivariant algebra, there are several definitions of Witt vectors, each corresponding to a definition of equivariant topological Hochschild homology (see [3], for instance).

1. What are the various types/definitions of equivariant Witt vectors?
2. Where do they show up in equivariant algebra?
3. Are any of these related to Hopf ring (like) structures appearing in equivariant Bredon homology?

VII. **Equivariant computations relating to Landweber exactness** (*Contributed by Sarah Petersen*)

A. Nonequivariantly, Dieudonné theory, together with Ravenel's computation of $BP_*\Omega^2 S^3$ [28], can be used to deduce the results of Hopkins–Hunton [17] and Hunton–Turner [19, 20] on the homology of the spaces representing a Landweber exact theory [12, Sections 10 and 11]. Equivariantly, in the $RO(C_2)$, or more generally $RO(G)$-graded setting, there is no obvious Dieudonné theory generalization, owing to the fact that maps between free $E_\star$-modules, where E is a G-ring spectrum may not have free $E_\star$-kernels or cokernels. Is there a way to algebraicize and/or categorify this?

B. Carrick–Guillou–Petersen compute $BP_{\mathbb{R}_\star}\Omega^\rho S^{\rho+1}$ (*in progress*), the C_2-equivariant analogue of $BP_*\Omega^2 S^3$. Thus we can ask what the equivariant computation of $BP_{\mathbb{R}_\star}\Omega^\rho S^{\rho+1}$ tells us, if anything, about the C_2-equivariant homology of C_2-equivariant analogues of Landweber exact spectra when the underlying nonequivariant computations are understood in the language of classical Dieudonné theory.

VIII. **Equivariant Brown–Gitler spectra** (*Contributed by Sarah Petersen*)

Classically, one of the main reasons it is practical to compute Hopf rings in homology by studying the associated ring object in Dieudonné modules is that there is close relationship between Brown–Gitler spectra and Dieudonné modules (see [12, 25, 24, 26]). It would be interesting to know to what extent a similar program can be carried out equivariantly.

A. One way of constructing Brown–Gitler spectra is via Brown representability and observing that the functor $E \to DH_*\Omega^\infty E$ taking a ring spectrum to the Dieudonné ring associated to the Hopf algebra $H_*\Omega^\infty E$ is exact (see [11, 10]). Similarly, in the C_2-equivariant world, is the functor $X \to DH_{*+b\sigma}\Omega^\infty X$ exact for each b? If it is exact, what are the properties of the represnting spectrum?

B. Classically, Brown–Gitler spectra can also be constructed from the Snaith summands of $\Omega^2 S^3$. This was proven by Brown–Peterson at the prime $p = 2$ [4], partially for odd primes by Ralph Cohen [8], and finally in all cases by Hunter–Kuhn [18]. In the C_2-equivariant setting, do the Snaith summands of $\Omega^\rho S^{\rho+1}$ deserve to be Brown–Gitler spectra? What are their comodules? Can we characterize them homotopically?

C. Non-equivariantly, two conditions characterize Brown-Gitler spectra. First, these spectra realize certain sub-comodules of the dual Steenrod algebra. Additionally, they satisfy a surjectivity condition coming from the geometry involved in Brown–Gitler's original construction and witnessed by a certain stage in their Postnikov construction. It is not currently known if a C_2-equivariant analogue of the surjectivity condition [5, Theorem 1.1 (ii)] holds or if there should be some other criterion determining (integral) Brown–Gitler spectra in the C_2-equivariant case. What homotopically characterizes C_2-equivariant Brown–Gitler spectra?

D. Can one construct C_2-equivariant Brown–Gitler spectra analogously to Brown and Gitler's original construction in [5]? Since the C_2-equivariant dual Steenrod algebra is a Hopf algebroid rather than a Hopf algebra, there is no obvious definition of the antipode s on the C_2-equivariant Steenrod algebra. What happens if one uses Brown–Comenetz duality to define a map playing the role of s? If such spectra are constructed, do they have useful computational properties? J.D. Quigley and others have been thinking about this problem, particularly in the motivic setting.

E. Note: David Chan and Sarah Petersen have work in progress using a Thom spectrum construction for C_p-equivariant Brown–Gitler spectra at all primes p.

3. Hopf algebras in the group cohomology of arithmetic groups

(*Contributed by Jeremy Miller and Peter Patzt*)

IX. Hopf rings and the cohomology of general linear groups
   A. Given a PID R, Ash–Miller–Patzt show that

$$\bigoplus_{n,k\geq 0} H_k(\mathrm{GL}_n(R);\mathrm{St}_n(F)),$$

   has the structure of a bigraded commutative Hopf algebra. Here F is the field of fractions of R and $\mathrm{St}_n(F)$ is the Steinberg module, the top reduced homology of the Tits building of F^n.
      1. In our paper, we conclude that rationally, it is a free graded commutative algebra. What can we deduce in characteristic $p > 0$? In particular, for $R = \mathbb{Z}$.
      2. We find wheel classes for $R = \mathbb{Z}$. Can we find analogs for other (number) rings R?
      3. Changing the general linear groups to certain subgroups should also give you a Hopf algebra structure, some of which may or may not be commutative. What can you deduce for those?

4. Hopf objects and broader structures

(*Contributed by Weinan Lin and Peter May*)

X. Hopf rings, E_∞-structures, and unstable operations (*Contributed by Weinan Lin and Peter May*)
   A. Given an E_∞ ring spectrum $\{E_n\}$ (using the Lewis-May model), what do all unstable multiplicative Dyer-Lashof operations $Q : H_*(E_n) \to H_*(E_{pn+q})$ look like? How do they interact with additive Dyer-Lashof operations? In $H_*(E_0)$ we have mixed Cartan formula and mixed Adem relations. Some of these unstable multiplicative operations will become stable Dyer-Lashof operations acting on the homology of the spectra. But the others will vanish.
   B. Tyler Lawson used secondary power operations to show that the 2-primary Brown–Peterson spectrum does not admit an E_n-structure for any $n \geq 12$. Are there primary operations in homology, perhaps visible in the full structure of Hopf rings, that one can use to deduce the same result?

XI. Hopf algebras as groups in the category of cocommutative coalgebras (*Contributed by Peter May*)
   A. In the category of cocommutative coalgebras, the tensor product is the cartesian monoidal product. Thus Hopf algebras that are cocommutative coalgebras are groups in this category. Thus we have notions such as fixed point groups and can seriously look at doing group theory in this setting. A paper by Guillou, May, and Rubin is a starting point for this perspective [13]. Initial questions include:
      1. Analyze the subgroups corresponding to finite subalgebras of the Steenrod algebra.
      2. Is there an algebraic chromatic homotopy theory defined by these subgroups, that is subgroups of the Steenrod algebra as a Hopf algebra?

5. Hopf Rings and Geometric Representatives

(*Contributed by Dev Sinha*)

XII. **The Geometry of Eilenberg–MacLane spaces**. The Geometry of EilenbergMacLane spaces. This is an open and ongoing a re- search project lead by Dev Sinha. Hanna Hoffman, Dana Hunter, Kristine Pelatt, Sarah Petersen and Courtney Thatcher have all contributed at various points. If you are interested in collaborating after the workshop, please email Dev Sinha at `dps@uoregon.edu`.

A. **Project description**. The goals of this project are to first elaborate Cartans computation of the homology of EilenbergMacLane spaces, giving a divided powers Hopf ring presentation of the homology, giving a basis in terms of graphs, and comparing all of these with both the original Cartan and RavenelWilson computations. Moreover, (geometric) chain representatives should be given. Conjecturally, the chains on any topological abelian group should have a divided powers structure, and conjecturally these are compatible with the circle multiplication in the simplest way possible. Developing these ideas in the mod-p setting would make for a nice paper, though mostly expository. Extending to the integer coefficient homology of Eilenberg-MacLane spaces for the integers could shed significant light on a calculation which is still not well understood. There, there is a conjecture that the divided powers structure governs (higher) torsion.

B. **Video lectures**.

- Geometry of Eilenberg-MacLane Spaces I - Homology Representatives (`https://www.youtube.com/watch?v=RYHoZDogBlU`)
- Geometry of Eilenberg-MacLane Spaces II - Iterated Bar (`https://www.youtube.com/watch?v=epxkF_LDjrQ`)
- Geometry of Eilenberg-MacLane Spaces III - Hopf ring structure and questions (`https://www.youtube.com/watch?v=x9YFLg_jVfg`)

XIII. Compute the Morava K-theory of symmetric groups. Recently, Guerra, Giusti, Salvatore and Sinha have, in a few different papers, computed cohomology of symmetric groups and more generally of extended powers as Hopf rings with divided powers. An interesting open question is to understand the associated Dieudonne algebras. The Hopf ring structure, originally developed by Strickland and Turner, is surprising not expected as Hopf rings are for homology of infinite loop spaces. Strickland considered this structure for Morava K-theory of symmetric groups, which remains a key open calculation in algebraic topology, and was able to calculate the indecomposibles with respect to the transfer product. With this work, the calculations for cohomology (which include full control of restriction maps to elementary abelian subgroups, and recent work of Hung on Dickson algebras all in place, the subject is ripe for progress.

6. Hopf rings and TMF

(*Contributed by W. Stephen Wilson*)

The following is a recruitment opportunity to work on a project involving Hopf rings and TMF. Ideal collaborators will have substation experience and familiarity working with TMF.

Abstract. There are a number of Hopf ring problems associated with TMF that I have been working on but do not have the expertise to finish or make rigorous. I am hoping to attract someone who knows what they are doing and who would like to help me out.

6.1. **The Hopf Ring for TMF at primes bigger than 3.** There is a lot of literature on tmf and I don't understand much of any of it, but I can pick out a few formulas and miscellaneous facts. For this section, I assume we are dealing with tmf and TMF localized at a prime, $p > 3$. I'll give no references because I'm trying to attract someone who knows it all already.

We are told by those who know such things, that

$$\pi_*(tmf) = \mathbb{Z}_{(p)}[c_4, c_6] \qquad |c_4| = 8 \quad |c_6| = 12$$

The same sort of people modify this using Δ, defined as

$$c_4^3 - c_6^2 = (12)^3\Delta$$

and then rewrite

$$\pi_*(tmf) = \mathbb{Z}_{(p)}[c_4, c_6, \Delta]/\big(c_4^3 - c_6^2 - (12)^3\Delta\big)$$

To me, since our prime is bigger than 3, that $(12)^3$ seems pointless since it is just a unit.

Then, to create TMF, all they seem to do is invert Δ. This seems to make TMF a 24-periodic Ω-spectrum.

Thus, for those of us who want to compute Hopf rings, we only have to compute it for 24 spaces here. Gotta be better than an infinite number of spaces.

However, for the cases we are working with, we are told a lot more. In particular, it seems that tmf is, for all practical purposes (and if we aren't practical, who is?), tmf is just a bunch of copies of $BP\langle 2\rangle$. This is why you don't want to compute the Hopf ring for all of tmf, see [32].

That's almost an embarrassing amount of information. If we didn't have it, we would still know, because we have been told, that our tmf is complex orientable. Among other things, this would mean we had a nice map

$$CP \longrightarrow \underline{TMF}_2$$

We can then take the homology generators of CP and map them into $\underline{TMF}_2$. This would put us in a familiar setting where we know the homotopy, and thus $H_0(\underline{TMF}_*)$ and these elements from CP. This should be everything. Nailing down the names of generators and such remains.

If we use the extra knowledge that we have because this is really nothing but a bunch of copies of $BP\langle 2\rangle$, we should be able to say a lot more.

In particular, we are told that the number of copies of $BP\langle 2\rangle$ is indexed by

$$\mathbb{Z}_{(p)}[c_4, c_6]/(p, v_1, v_2)$$

I'm more than a little unclear on what v_1 is here, but we are told

$$v_2 = (-1)^{\frac{p-1}{2}}\Delta^{\frac{p^2-1}{12}} \qquad \mathrm{mod}(p, v_1)$$

It is also pointed out that $\mathbb{Z}_{(p)}[c_4, c_6]/(p, v_1, v_2)$ is finite and bounded above by $2(p^2 + p - 12)$.

Now, since v_2 is almost a multiple of Δ (this is why, among other reasons, I need help), if we invert Δ, we have inverted v_2.

Now, if we were back in the real world (as I know it), and only looking at $BP\langle 2\rangle$, and we inverted v_2, we would get the so-called Johnson-Wilson spectrum $E(2)$. This would now be v_2 periodic, i.e. have period $2(p^2 - 1)$. Furthermore, its homology is well understood. It is easy to give a basis for the algebra generators etc. You can just modify "allowable" in [29] knowing that $[v_2]$ is always around.

I presume (yet another reason I need help) that the number of copies of $BP\langle 2\rangle$ listed above is what allows this to be reduced to period 24. But this shouldn't much alter the homology, just mess around with the bookkeeping a bit (more help please).

The point here is that the Hopf ring $H_*(\underline{TMF}_*)$ should be easily within our grasp for primes bigger than 3.

If, for some reason, they are harder then they should be, restricting to $p = 5$ would already be interesting. (More so if the others are harder.)

In the case of $p = 5$, c_4 has the same degree as v_1, suggesting that they might be one and the same. Furthermore, we already know that v_2 is approximately Δ^2. There are presumably not many copies of $BP\langle 2\rangle$ involved here and it should be not so hard to get things straight for $p = 5$ even if bigger primes put up more resistance.

I am clearly in need of help, and welcome it.

6.2. **Now for something interesting, the prime 3.** The problem with the prime 3 is that we no longer have complex orientability. Thus, names are hard to give. Maybe one of you knows some space where we have a map of it into some $\underline{TMF}_k$ and can get some named elements that way? Maybe a lens space?

With or without such, we do know the homotopy, and it isn't that complicated. Furthermore, TMF is only 72 periodic. There is hope there. If we knew the homology of all 72 spaces, then it would probably be straightforward to prove. I have done some computations and at the time of writing I have the first 23 homology groups of $\underline{TMF}_0$ for example. As yet, I have not run into any obstacles I couldn't overcome. The names are starting to get unpleasant though.

6.3. **Now for something else entirely, the prime 2.** I spent an entire year of the pandemic computing lots and lots of the homology of the Ω-spectrum for tmf at $p = 2$. It was fun. It was frustrating. It got nowhere. As for TMF, it is something like 576 periodic, and the homotopy isn't all that nice. Starting at $p = 2$ was just plain stupid.

I talked (talk=email) with Pearson a lot about this problem as he had been there before me.

The one thing that came out of it is the conjecture that the map

$$H_*(QS^0) \longrightarrow H_*(\underline{tmf}_0)$$

is surjective. That means that $H_*(\underline{tmf}_0)$ can be written in terms of homology operations on the class [1] in degree 0. Despite lots of information, no conjecture about the homology operations needed showed up. If one could guess the correct answer, then it might be possible to compute lots of things. As is stands, the input information is $\pi_*(tmf)$ and $H^*(tmf)$. In principle, that's all it takes. That an an infinite number of monkeys making random computations for an infinite amount of time.

More than you want to know.

For $p = 2$, starting with the zero degree homology we get from the very complicated $\pi_*(tmf)$, is unlikely to be done by anyone, no matter their tolerance for pain. However, if we really did get a grip on $H_*(\underline{tmf}_0)$, then one could go after the positive spaces in a reasonable way.

This would be similar to what was done for BO in [23]. Although we know the stable $H^*(tmf)$, it is not written down in a way that is relevant to Ω-spectrum computations. Similarly, the known $H^*(bo)$ in terms of the anti-automorphism of Milnor basis elements didn't work, nor would the new Cartan-Serre type basis in [9]. A very different basis for $H^*(bo)$ comes out of [23] and the same sort of non-standard basis for $H^*(tmf)$ would come out of a similar type of computation for the positive spaces for tmf. However, if one really had $H_*(\underline{tmf}_0)$, it would be possible.

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