Equivariant Steenrod Operations

BY

PRASIT BHATTACHARYA

UNIVERSITY OF NOTRE DAME

joint work with

MINGCONG ZENG

Throughout this talk

- $p = 2$
- G – a finite 2-group

Classical Steenrod Operations

Norman Steenrod (1947)

- For any space X

$$\boxed{Sq^i : H^n(X;\mathbb{F}_2) \longrightarrow H^{n+i}(X;\mathbb{F}_2).}$$

- Sq^i is natural in X
- $Sq^i(x) = \begin{cases} 0 & n < i \\ x^2 & n = i \end{cases}$
- $Sq^i(\sigma^*(u)) = \sigma^* Sq^i(u)$ $\qquad \sigma^* : H^{*+1}(\Sigma X) \cong H^*(X)$

In 1952, José Adem proved a conjecture of Wen-tsün Wu

$$\boxed{Sq^i Sq^j = \sum_{k=0}^{\lfloor i/2 \rfloor} Sq^{i+j-k} Sq^k \quad \text{when } 2i<j}$$

Steenrod Algebra

$$A = \frac{\mathbb{F}_2\{Sq^i : i \in \mathbb{N}\}}{\langle \text{Adem Relations} \rangle}$$

Modern Reformulation

$$\boxed{H^n(X;\mathbb{F}_2) \cong [X, \Sigma^n H\mathbb{F}_2] = [X, H\mathbb{F}_2]_{-n}}$$

- $H\mathbb{F}_2$ – mod 2 Eilenberg McLane spectrum.

$$\boxed{\mathcal{A} \cong [H\mathbb{F}_2, H\mathbb{F}_2]_*}$$

Notable Applications

(1) Lead to the construction of Stiefel Whitney classes.

Thom isomorphism,

$$\boxed{w_i = \tau^{-1} Sq^i(u)}$$

Thom class.

(2) Lead to Adams spectral sequence

$$\boxed{\mathrm{Ext}_A(\mathbb{F}_2, \mathbb{F}_2) \Rightarrow \pi_*(\mathbb{S}) \otimes \mathbb{Z}_2}$$

(3), (4), ...

Equivariant Steenrod Algebra

(an abstract definition)

- For any G-space or spectra X,

$$\boxed{H_G^{\star}(X;\underline{\mathbb{F}_2}) \cong [X, H\underline{\mathbb{F}_2}]^G_{-\star}, \quad \star \in RO(G)}$$

$H\underline{\mathbb{F}_2}$ - G-equivariant Eilenberg spectrum with coefficient in $\underline{\mathbb{F}_2}$

- G-equivariant Steenrod algebra.

$$\boxed{A^G := [H\underline{\mathbb{F}_2}, H\underline{\mathbb{F}_2}]^G_{\star}}$$

⊙ Note $A^G := [H\underline{\mathbb{F}}_2, H\underline{\mathbb{F}}_2]^G$ is an algebra over

$$\boxed{\mathbb{M}_G := [\mathbb{S}_G, H\underline{\mathbb{F}}_2]^G.}$$

Known Results

- $G = 1$: (Steenrod (1947), Adem (1952))
- $G = C_2$: $\mathbb{M}_{C_2} = \mathbb{F}_2[a_\sigma, u_\sigma] \oplus \theta\{a_\sigma^{-i} u_\sigma^{-j} ; i,j \geq 0\}$, and

$$\boxed{\mathcal{A}^{C_2} := \frac{\mathbb{M}_{C_2}[\, Sq^i \,;\, i \geq 0 \,]}{(\text{Adem Relations})}}$$

Hu-Kriz (2001)
Voevodsky (2003)

- $\mathbb{M}_G$ is not known for $G \neq \underline{e}, C_2, C_4$!!!

Hu-Kriz

Mingcong
Nick G

Today's Goal

- Construct G-equivariant squaring operations

$$\boxed{Sq^{\alpha} : H^{\star}_{G}(-;\underline{\mathbb{F}}_2) \longrightarrow H^{\star+\|\alpha\|}(-;\underline{\mathbb{F}}_2)}$$

where α is an EULERIAN SEQUENCE.

A G-equivariant tautological bundle

- Recall, the tautological line bundle over $\mathbb{R}P^\infty$

$$\begin{array}{c} E(\gamma_1) := \operatorname*{colim}_{n\to\infty} S(n\tau) \times_{\Sigma_2} (\tau) \\ \big\downarrow \\ \mathbb{R}P^\infty = B\Sigma_2 := \operatorname*{colim}_{n\to\infty} S(n\tau) \times_{\Sigma_2} * \end{array}$$

τ - sign representation of Σ_2

- Let ρ denote the regular representation of G.
- Let τ denote the sign representation of Σ_2.

⊙ The G-equivariant bundle γ_G :

forget G-action.

$$|G|\gamma_1 \leftarrow E(\gamma_G) := \operatorname*{colim}_{n\to\infty} S(n\rho\otimes\tau) \times_{\Sigma_2} (\rho\otimes\tau)$$

$$\downarrow$$

$$B_G\Sigma_2 := \operatorname*{colim}_{n\to\infty} S(\underbrace{n\rho\otimes\tau}_{G\times\Sigma_2 \text{ rep}^n}) \times_{\Sigma_2} (0)$$

Claim: γ_G is $H\underline{\mathbb{F}_2}$-orientable!

Proof: $H\underline{\mathbb{F}_2}$ does not distinguish between

$$\gamma_G : \boxed{\begin{array}{c} \operatorname{colim}_{n\to\infty} S(n\rho\otimes\tau) \times_{\Sigma_2} (\rho\otimes\tau) \\ \downarrow \\ \operatorname{colim}_{n\to\infty} S(n\rho\otimes\tau) \times_{\Sigma_2} (0) \end{array}}$$

and

$$\varepsilon_G : \boxed{\begin{array}{c} \operatorname{colim}_{n\to\infty} S(n\rho\otimes\tau) \times_{\Sigma_2} (\rho) \\ \downarrow \\ \operatorname{colim}_{n\to\infty} S(n\rho\otimes\tau) \times_{\Sigma_2} (0) \end{array}}$$

$$\underline{\mathbb{F}_2}^\times = \underline{1}$$

Consequently:

- $H\underline{\mathbb{F}}_2$ - Thom isomorphism.

$$\boxed{\begin{aligned} T(\gamma_G)\wedge H\underline{\mathbb{F}}_2 &\underset{G}{\simeq} T(\varepsilon_G)\wedge H\underline{\mathbb{F}}_2 \\ &\underset{G}{\simeq} \Sigma^{\rho} B_G\Sigma_{2+}\wedge H\underline{\mathbb{F}}_2 \end{aligned}}$$

- $H\underline{\mathbb{F}}_2$ - Thom class, $u \in H^{\rho}_G(T(\gamma_G);\underline{\mathbb{F}}_2)$.

- $H\underline{\mathbb{F}}_2$ - Euler class. $\widehat{e} \in H^{\rho}_G(B_G\Sigma_{2+};\underline{\mathbb{F}}_2)$.

zerosection: $B_G\Sigma_{2+} \longrightarrow T(\gamma_G)$

Defn : A sequence of elements in $H_*^G(B_G\Sigma_2;\mathbb{F}_2)$

$$\alpha = (a_0, a_1, a_2, \ldots\ldots)$$

is called EULERIAN if

$$\boxed{\begin{array}{l} \text{(i)}\ \ a_{i+1} \cap e = a_i \\ \text{(ii)}\ \ a_0 \cap e = 0 \end{array}}$$

Remark

(i) $|a_i| = p + |a_{i-1}| = \cdots = ip + |a_0|$

$$\boxed{\|\alpha\| = -|a_0| = ip - |a_i|}$$

(ii) If α is Eulerian then so is

$$\boxed{\alpha[n] = (\underbrace{0, \cdots, 0}_{n\text{-many}}, a_0, a_1, a_2, \cdots)}$$

Examples

(i) $\boxed{G=1}$; $H_*(B\Sigma_{2+}) = \mathbb{F}_2\{b_0, b_1, b_2, \ldots\}$

where b_i is dual to $t^i \in H^*(B\Sigma_{2+}) \cong \mathbb{F}_2[t]$

- EULER CLASS : $e = t$
- $b_{i+1} \cap e = b_i$
- Thus, $\beta = (b_0, b_1, \ldots)$ is an Eulerian sequence

$$\boxed{Sq^{\beta} = 1}$$

- $\beta[n] = (0, 0, \ldots, 0, b_0, b_1, \ldots)$

$$\boxed{Sq^{\beta[n]} = Sq^{n}.}$$

(ii) $\boxed{G = C_2}$ $\qquad a_\sigma, u_\sigma \in \mathbb{M}_{C_2}$

Hu-Kriz
Voevodsky

$$\boxed{H^{\star}_{C_2}(B_{C_2}\Sigma_{2+}) \cong \frac{\mathbb{M}_{C_2}[y, x]}{(y^2 = a_\sigma y + u_\sigma x)}}$$

$|y| = \sigma$
$|x| = \rho$

- $H^{\star}_{C_2}(B_{C_2}\Sigma_{2+}) = \mathbb{M}_{C_2}\{y, x, yx, x^2, yx^2, \ldots\}$
- $H_{\star}^{C_2}(B_{C_2}\Sigma_{2+}) = \mathbb{M}_{C_2}\{b_\sigma, b_\rho, b_{\rho+\sigma}, b_{2\rho}, b_{2\rho+\sigma}, \ldots\}$

Euler class of γ_{C_2} is x

- Two families of Eulerian sequence....

- $\beta[n] := \{0, \ldots, 0, 1, b_\rho, b_{2\rho}, \ldots\}$.

- $\gamma[n] := \{0, \ldots, 0, b_\sigma, b_{\rho+\sigma}, b_{2\rho+\sigma}, \ldots\}$

$$\boxed{Sq^{\beta[n]} = Sq^{2^i}} \qquad \boxed{Sq^{\gamma[n]} = Sq^{2^i+1}}$$

Philosophy / Key idea.

Classical Steenrod operation is a consequence of two geometric facts:

(i) $H\mathbb{F}_2$ is an $\mathbb{E}_\infty$-ring spectrum. $D_n(E) = E\Sigma_{n+} \wedge_{\Sigma_n} E^{\wedge n}$

Σ_n by perm.

$$\theta_n : D_n(H\mathbb{F}_2) \longrightarrow H\mathbb{F}_2$$

(ii) The tautological line bundle γ_1 is $H\mathbb{F}_2$-orientable.

$$u_n : T(n\gamma_1) \longrightarrow \Sigma^n H\mathbb{F}_2.$$

Step 1 (i)+(ii) $\Longrightarrow$ Power Operation

If $u \in H^n(X_+;\mathbb{F}_2)$ i.e. $u : X_+ \longrightarrow \Sigma^n H\mathbb{F}_2$

$$P(u) \in H^{2n}(D_2(X_+);\mathbb{F}_2)$$

$D_2(X_+) = \left(E\Sigma_2 \times_{\Sigma_2} X^{\times 2}\right)_+$

Abbreviate: $H := H\mathbb{F}_2$

$$\begin{tikzcd}
D_2(X_+) \arrow[d, "D_2(u)"'] \arrow[rrrr, bend left, "P(u)"] & & & & \\
D_2(\Sigma^n H) \arrow[r] & D_2(S^n) \wedge D_2(H) & & & \\
& \Sigma^n T(n\gamma_1) \wedge D_2(H) \arrow[u, phantom, "\cong" description] \arrow[r, "\Sigma^n u_n \wedge \theta_2"] & \Sigma^{2n} H \wedge H \arrow[r] & \Sigma^{2n} H &
\end{tikzcd}$$

$D_2(S^n) = E\Sigma_2 \times_{\Sigma_2} (S^n \times S^n) = \Sigma^n T(n\gamma_1)$

Power Operation $\Rightarrow$ Squaring Operations

- The diagonal map

$$\begin{array}{ccc} X & \longrightarrow & X \times X \\ \circlearrowleft & & \circlearrowleft \\ \text{trivial} & & \text{flip} \end{array}$$

is Σ_2-equivariant and induces a map

$$\delta : \underset{E\Sigma_2 \times_{\Sigma_2} X}{\underset{\shortparallel}{B\Sigma_2 \times X}} \longrightarrow D_2(X) = E\Sigma_2 \times_{\Sigma_2} X^{\times 2}$$

- The following formula defines the Steenrod squaring operations.

$$\boxed{\delta^* P(u) = \sum_{i=-n}^{n} Sq^i(u) \otimes t^{n-i}}$$

$$H^*(B\Sigma_2) \otimes H^*(X_+) \cong H^*(B\Sigma_{2+} \wedge X_+)$$

Kunneth iso for

The G-equiv Steenrod operation is a consequence of two geometric facts:

(i) $H\underline{\mathbb{F}_2}$ is an $\mathbb{E}_\infty^G$-ring spectrum.

$$\theta_n : D_n^G(H\underline{\mathbb{F}_2}) \longrightarrow H\underline{\mathbb{F}_2}$$

where, $D_n^G(E) = E_G\Sigma_{n_+} \underset{\Sigma_n}{\wedge} E^{\wedge n}$

- $E_G\Sigma_n$ is a $G\times\Sigma_n$-space defined by the universal property

$$E_G\Sigma_n^{\,H} \simeq \begin{cases} * & H\subset G\times\Sigma_n \text{ st } H\cap\Sigma_n = 1 \\ \phi & H\subset G\times\Sigma_n \text{ st } H\cap\Sigma_n \neq 1. \end{cases}$$

(ii) γ_G is $\underline{H\mathbb{F}_2}$-orientable.

$$\boxed{\Rightarrow u_n : T(n\gamma_G) \longrightarrow \Sigma^{np}\underline{H\mathbb{F}_2}}$$

(i) + (ii) $\Rightarrow$ Power operation

$$\boxed{\mathcal{P}(-): H_G^{np}(-) \longrightarrow H_G^{2np}(D_2^G(-))}$$

$u \in H^{np}(X_+)$.

Abbrev $H := H\underline{\mathbb{F}_2}$

$$\begin{tikzcd}
D_2^G(X_+) \arrow[d, "D_2^G(u)"] \arrow[rrrr, bend left, "P(u)"] & & & & \\
D_2^G(\Sigma^{np} H) \arrow[r] & D_2^G(S^{np}) \wedge D_2^G(H) \arrow[d, phantom, "\wr"] & & & \\
& \Sigma^{np} T(n\gamma_G) \wedge D_2^G(H) \arrow[rr, "\Sigma^{np} u_n \wedge \theta_2"] & & \Sigma^{2np} H \wedge H \arrow[r, "\mu"] & \Sigma^{2np} H
\end{tikzcd}$$

If X is a G-space, then the diagonal map

$$\Delta : \underset{\substack{\circlearrowleft \\ \text{trivial action} \\ \text{of } \Sigma_2}}{X} \longrightarrow \underset{\substack{\circlearrowleft \\ \text{flip action of} \\ \Sigma_2 .}}{X \times X}$$

is $G \times \Sigma_2$-equivariant map.

$\Rightarrow$ The map Δ induces a G-equiv map.

$$\boxed{\delta : B_G \Sigma_{2_+} \wedge X_+ \longrightarrow D_2^G(X_+)}$$

If $u \in H_G^{np}(X_+)$ then we have:

(i) $P(u) \in H_G^{2np}(X_+)$.

(ii) $\delta^* P(u) \in H_G^{2np}(B_G \Sigma_{2+} \wedge X_+)$

But we do not have a Kunneth isomorphism!

G-equivariant Steenrod Operations

<u>Def</u>

Defn: Let $\alpha = (a_0, a_1, a_2, \ldots)$ be an Eulerian sequence. For $u \in H^{np}(X_+)$ define

$$\boxed{Sq^{\alpha}(u) = u / a_n.}$$

$(-)/(-) : H^*(A \times B) \times H_*(A) \longrightarrow H^*(B)$ is the slant product.

Conjecture:

$$\boxed{\{Sq^{\alpha} : \alpha \text{ is Eulerian}\} \text{ generates } \mathcal{A}^G := [H\underline{\mathbb{F}}_2, H\underline{\mathbb{F}}_2]^G_\star}$$

- Total Squaring Operations?

- Adem Relations?

FUTURE GOAL (I)

- Identify Eulerian sequences in $H_*^G(B_G\Sigma_{2+};\underline{\mathbb{F}_2})$

Theorem/Conjecture

- Let $e \in H^{\rho}_{G}(B_G\Sigma_{2+})$ denote the Euler class.

- There exists

$$\boxed{b_n \in H^{G}_{n\rho}(B_G\Sigma_{2+})}$$

dual to $e^{n} \in H_{G}(B_G\Sigma_{2+})$ s.t. $\beta[n] = (0,\dots,0,1,b_1,b_2,\dots)$ is Eulerian.

$$\beta[n] \overset{\text{leads}}{\leftrightsquigarrow} Sq^{n\rho}_{G}$$

- Let G be an Abelian group and $H < G$

$$\mathrm{Res}^{H}_{*} : H^{*}_{G}(X) \longrightarrow H^{*}_{H}(\mathrm{Res}^{H}(X))$$

$$\hat{\Phi}^{H}_{*} : H^{*}_{G}(X) \longrightarrow H^{*}_{G/H}(\Phi^{H}(X))$$

- $\mathrm{Res}^H_*\left(Sq^{np}_G(u)\right) = Sq^{n[G:H]p}_H\left(\mathrm{Res}^H_*(u)\right)$
- $\hat{\Phi}^H_*\left(Sq^{np}_G(u)\right) = Sq^{np}_{G/H}\left(\hat{\Phi}^H_*(u)\right)$

FUTURE GOAL (II)

Potential applications problems in geometry

- Let W and V be G-representations.

  $P(W) = \{\text{Lines in } W\}$.

Question: Does there exists a G-equivariant immersion $P(W) \hookrightarrow V$?

- Cobordism ring of G-equivariant manifolds ?

THE END