Computing Differentials in the Adams Spectral Sequence

joint work w/ Weinan Lin and Guozhen Wang

<u>Question</u>: Does there exist $n \times n$ real matrices

$A_1, A_2, \cdots, A_n$ s.t. $\forall r \neq 0$ in $\mathbb{R}^n$

$A_1 r, A_2 r, \cdots, A_n r$ are always linear independent?

<u>Answer</u>: $\Leftrightarrow$ Hopf inv. 1 problem

i.e. $TS^{n-1} \cong S^{n-1} \times \mathbb{R}^{n-1}$ $\overset{\text{Adams}}{\Leftrightarrow}$ $n = 1, 2, 4, 8$.

$\Leftrightarrow$ $f: S^{2n-1} \longrightarrow S^n$

$H^*(Cf ; \mathbb{Z}/2) \cong \mathbb{Z}/2[x_n]/x_n^3$ $\deg x_n = n$.

<u>Rmk</u>:
$\mathbb{R}P^2, \mathbb{C}P^2, \mathbb{H}P^2, \mathbb{O}P^2$

$$Sq^n \left[\begin{array}{ll} \circ & 2n \\ \circ & n \end{array} \right.$$

- $Sq^n x_n = x_n^2$

$\Rightarrow$ Sq^n is indecomposable in the Steenrod alg A.

<u>Adem</u>: $\Rightarrow$ $n = 2^m$ for some m.

<u>Adams</u>: Sq^{2^m}, $m \geq 4$, is decomposable in terms of secondary cohom operations

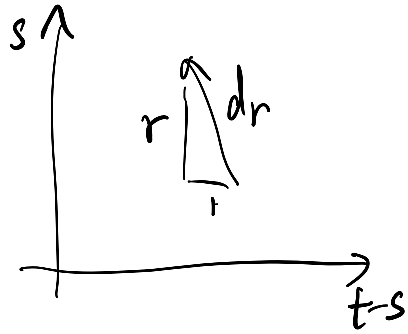
$$Sq^{16} = Sq^1 \Phi_{3,3} + \text{other decomposable terms}$$

$\Phi_{3,3}$: secondary operation assoc. w/.

$$Sq^8 Sq^8 = Sq^{12} Sq^4 + Sq^{14} Sq^2 + Sq^{15} Sq^1$$

The Adams SS $\quad p=2$

$$E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \pi_{t-s} \hat{S}^0$$



$$d_r : E_r^{s,t} \longrightarrow E_r^{s+r,\, t+r-1}$$

Adams 1-line: $\mathrm{Ext}_A^{1,t} = \begin{cases} \mathbb{F}_2 & \text{if } t = 2^j \\ 0 & \text{o/w} \end{cases}$

gen. by $h_j \longleftrightarrow [Sq^{2^j}]$ indecomposable in A.

Theorem (Adams)

$$d_2(h_j) = h_0 \cdot h_{j-1}^2 \neq 0 \qquad \forall j \geq 4$$

$$Sq^{2^j} = Sq^1 \cdot \Phi_{j-1,j-1} + \cdots$$

$$\Phi_{j-1,j-1} \longleftrightarrow Sq^{2^{j-1}} Sq^{2^{j-1}} + \cdots$$

- Baues, Jibladze, Nassau : secondary Steenrod alg. $\boxed{1}$

  Chua, Lin : Adams d_2 differentials

- $d_2(h_4) = h_0 h_3^2$

h_3 survives and detects $\sigma \in \pi_7$

$\sigma \cdot \sigma = -\sigma \cdot \sigma \Rightarrow 2\sigma^2 = 0 \Rightarrow h_0 h_3^2$ must be killed.

$\boxed{2}$ homotopy relations

$\boxed{3}$ The Leibniz rule.

- $d_2(h_1 h_3) = 0$

- $d_2(h_5) = h_0 h_4^2$

$$h_j h_{j+1} = 0 \quad \forall j \quad \Rightarrow h_4 h_5 = 0$$

$$\Rightarrow \quad 0 = d_2(h_4 h_5) = d_2(h_4)\cdot h_5 + h_4 \cdot d_2(h_5)$$
$$= h_0 h_3^2 \cdot h_5 + h_4 \cdot d_2(h_5)$$

$$h_0 h_3^2 \cdot h_5 = h_0 h_4^3 \neq 0 \qquad \Rightarrow d_2(h_5) = h_0 h_4^2 \neq 0.$$

- $d_3(h_0 h_4) = h_0 d_0$

$\boxed{4}$ Comparison w/ Toda's unstable computation

$$\pi_{14} = \mathbb{Z}/2 \oplus \mathbb{Z}/2$$

$\boxed{5}$ Comparison w/ J

$$\pi_{15} J = \mathbb{Z}/32$$

<u>Rmk</u>: $\boxed{3}\boxed{4}\boxed{5}$ May: up to 28-stem

$\boxed{6}$ Moss's Thm: Massey product $\longleftrightarrow$ Toda brackets

- $h_1 h_4 = \langle h_1, h_0, h_3^2 \rangle \rightsquigarrow \eta_4 \in \langle \eta, 2, \sigma^2 \rangle$

  h_4

  "no crossing diff"

$\Rightarrow d_r(h_1 h_4) = 0 \quad \forall r.$

$$\begin{tikzcd} \circ \arrow[r, "\sigma^2"] & \circ \arrow[d, dash, "2"'] & \\ & \circ \arrow[r, "\eta"'] & \circ \end{tikzcd}$$

- $h_4^2 = \langle h_0, h_3^2, h_0, h_3^2 \rangle \rightsquigarrow \langle 2, \sigma^2, 2, \sigma^2 \rangle$

$\qquad h_4 \quad h_4 \quad h_4$

$\qquad\quad 0 \quad 0$

$\Rightarrow d_r(h_4^2) = 0 \quad \forall r.$

$$\begin{tikzcd}
o \arrow[r, "\sigma^2"] & o \arrow[d, dash, "2"'] & \\
& o \arrow[d, dash, "\sigma^2"'] & \\
& o \arrow[r, "2"'] & o
\end{tikzcd}$$

<u>Rmk</u>: $\boxed{1}\boxed{2}\boxed{5}\boxed{6} \Rightarrow d_3(\overset{r}{\overset{\shortparallel}{\Delta h_2^2}}) = h_1 d_0^2$.

$\boxed{7}$ Mahowald's Thm through Brown-Gitler spectra.

$d_r(h_1 h_j) = 0 \quad \forall r, \forall j.$

- <u>Mahowald Uncertainty Principles</u>

  - <u>1st Principle</u>:

    Any spectral sequence $\Rightarrow \pi_* S^0$ w/ $E_2 \cong \mathrm{Ext}_?$ has ∞ many nonzero differentials.

  - <u>2nd Principle</u>:

    Any method that computes these nonzero differentials will leave ∞ many differentials undecided.

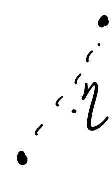
18 The Mahowald trick : diffs $\longleftrightarrow$ ext'ns.

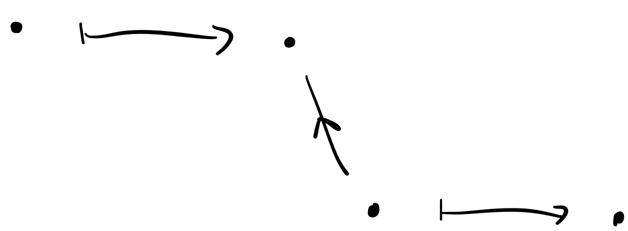
- $S^1 \xrightarrow{\eta} S^0 \longrightarrow C\eta \longrightarrow S^2 = \Sigma S^1$

  $H\mathbb{F}_{2*} = 0$

$$\Rightarrow \quad 0 \to H\mathbb{F}_{2*} S^0 \longrightarrow H\mathbb{F}_{2*} C\eta \longrightarrow H\mathbb{F}_{2*} S^2 \longrightarrow 0$$

$$\Rightarrow \quad \begin{tikzcd} {} \arrow[r, "\cdot h_1"] & \mathrm{Ext}(S^0) \arrow[r] \arrow[d, Rightarrow] & \mathrm{Ext}(C\eta) \arrow[r] \arrow[d, Rightarrow] & \mathrm{Ext}(S^2) \arrow[r, "\cdot h_1"] \arrow[d, Rightarrow] & {} \\ {} \arrow[r, "\cdot \eta"] & \pi_* S^0 \arrow[r] & \pi_* C\eta \arrow[r] & \pi_* S^2 \arrow[r, "\cdot \eta"] & {} \end{tikzcd}$$





Rmk : Barratt – Mahowald – Tangora : 29 – 45

$\boxed{9}$ Bruner's power operation

- $d_2(e_0) = h_1^2 d_0 \Rightarrow d_3(e_1) = h_1 t$

- $d_r(h_1 h_j) = 0 \quad \forall r, \forall j \Rightarrow d_r(h_2 h_{j+1}^2) = 0 \quad \forall r, \forall j$.

$\boxed{10}$ Comparison w/ tmf

$\Rightarrow g^3$ is <u>not</u> killed by any differential.

$\boxed{11}$ Comparison w/ the motivic Adams SS by Isaksen.

$$S^{0,0} \xrightarrow{\text{Betti Re}} S^0$$

$$\text{mot Adams SS}(S^{0,0}) \longrightarrow \text{Adams SS}(S^0)$$

$$s,t,w \longmapsto s,t$$

$$d_r \longmapsto d_r$$

$\rightsquigarrow$ • $d_3(Q_2) = gt$

+ rigorous arguments up to 59-stem

12 Motivic higher Leibniz rule by Xu

• $d_r(ab) = d_r a \cdot b + a \cdot d_r b$

• $d_r\langle a, b, c\rangle \in \langle d_r a, b, c\rangle + \langle a, d_r b, c\rangle + \langle a, b, d_r c\rangle$

$\Rightarrow d_2(D_1) = h_0^2 h_3 g_2$

13 Comparison w/ unstable Adams SS by Balderrama

[4] $\mathbb{RP}^\infty$-method by Wang–Xu.

- Kahn–Priddy $\mathbb{RP}^\infty \xrightarrow{KP} S^0$ surj. on $\pi_{*>0}$

- Lin. $\mathrm{Ext}(\mathbb{RP}^\infty) \longrightarrow \mathrm{Ext}(S^0)$ surj on $\mathrm{Ext}^{\mathrm{stem}>0}$

$\rightsquigarrow$ study subquotients of $\mathbb{RP}^\infty$

$\Rightarrow$ $d_3(D_3) = B_3$

$+$ 2 other differentials

$\rightsquigarrow$ 61-sphere has a unique smooth structure.

$\boxed{15}$ The motivic cofiber of τ method by Isaksen-Wang-Xu

$$\begin{array}{ccccc} SH^{\wedge} & \xleftarrow{\tau^{-1}} & \hat{S}^{0,0}-\mathrm{Mod} & \xrightarrow{\mathrm{mod}\ \tau} & \hat{S}^{0,0}/\tau-\mathrm{Mod} \\ & & & & \wr\!\text{|} \\ & & & & \mathcal{D}(BP_*BP-\mathrm{Comod}) \end{array}$$

Gheorghe-Wang-Xu.

$$\begin{array}{ccccc} \mathrm{ASS}(\hat{S}^0) & \xleftarrow{\tau^{-1}} & \mathrm{mot}\,\mathrm{ASS}(\hat{S}^{0,0}) & \xrightarrow{\mathrm{mod}\ \tau} & \mathrm{mot}\,\mathrm{ASS}(\hat{S}^{0,0}/\tau) \\ & & & & \wr\!\text{|} \\ & & & & \mathrm{alg\ Novikov SS}(BP_*) \end{array}$$

$\rightsquigarrow$ many diffs $\quad$ Ex $\quad d_4(h_5^3) = h_0^3 g_3 \neq 0$

$\rightsquigarrow$ almost all differentials up to 90-stem.

$\boxed{16}$ Comparison w/ synthetic Adams SS by Burklund-Isaksen-Xu.

$$\Rightarrow d_9(h_6 g + h_2 e_2)^{(83,5)} = 0$$

Rmk: $\boxed{15}+\boxed{16}$ Burklund-Xu:

$$d_4(h_j^3) = h_0^3 g_{j-2} \neq 0 \qquad \forall j \geq 6$$

Pstragowski, Gheorghe-Isaksen-Krause-Ricka

$$SH^\wedge \xleftarrow{\lambda^{-1}} \hat{S}^{0,0}\text{-Mod} \xrightarrow{\bmod \lambda} D(A_*\text{-Comod})$$

- $\pi_{*,*} S/\lambda \cong Ext_A^{*,*}$
- syn ASS is rigid.
  - λ-bockstein SS
  - syn ASS $\longleftrightarrow$ ASS

  $Ext_A[\lambda]$ $\qquad$ Ext_A

$d_r x = \lambda^{r-1} y$ $\qquad\qquad$ $d_r x = y$

Ex $\quad d_2(h_4) = h_0 h_3^2$, $\quad d_3(h_0 h_4) = h_0 d_0$

syn: $\quad d_2(h_4) = \lambda h_0 h_3^2$, $\quad d_3(h_0 h_4) = \lambda^2 h_0 d_0$

S/λ $\quad d_2(h_4) = 0$, $\quad d_3(h_0 h_4) = 0$

S/λ^2 $\quad d_2(h_4) = \lambda h_0 h_3^2$, $\quad d_3(h_0 h_4) = 0$

$\qquad d_2(\lambda h_4) = 0$

- $\pi_{*,*} S/\lambda^n \longleftrightarrow$ Adams E_{n+1}-page.

- $S \xrightarrow{\lambda^n} S \longrightarrow S/\lambda^n \longrightarrow \Sigma^{1,-n} S$

total Adams diff.

Ex $n=1$, on E_∞

$h_4 \longmapsto h_0 h_3^2$

$\rightsquigarrow$ • define ext'n on the Adams E_n-page

• translate diff to ext'n.

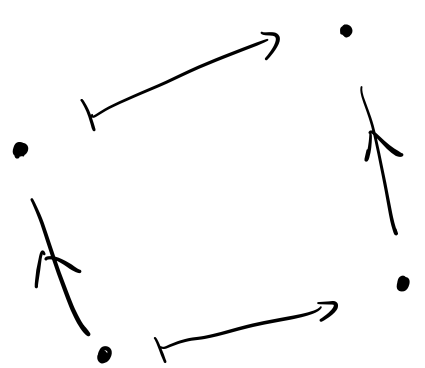
With certain "no crossing ext'n/diff" conditions.

$\boxed{17}$ Generalized Leibniz rule

$\boxed{18}$ Generalized Mahowald trick

Rmk: Behrens, Burklund, Chua, Ma, etc.

$\boxed{17}$ $X \longrightarrow Y$



$$\begin{array}{ccc} X/\lambda^n & \longrightarrow & Y/\lambda^n \\ \delta\downarrow & & \downarrow\delta \\ \Sigma^{1,-n}X & \longrightarrow & \Sigma^{1,-n}Y \end{array}$$

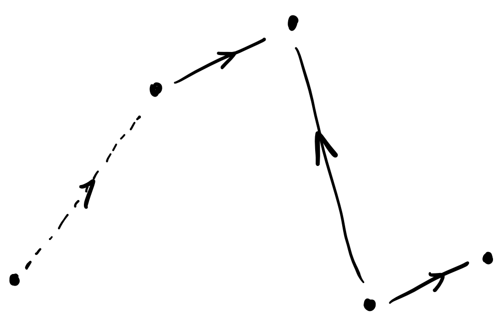
Ex $d_2(h_5) = \lambda h_0 h_4^2 \quad \Rightarrow \quad d_3(h_2 h_5) = \lambda^2 h_0 p$

ν-ext'n from $h_0 h_4^2$ to $\lambda h_0 p$

$\Uparrow$

in S/ν, $d_2(h_0 h_4^2[4]) = \lambda h_0 p[0]$

$\boxed{18}$ $X \to Y \to Z \to \Sigma X$



Theorem (Lin-Wang-Xu)

h_6^2 survives in the Adams SS.

Browder's Theorem $\Rightarrow$ There exists framed manifolds w/ Kervaire inv. 1 in dim 126.

Sketch proof: • Barratt-Jones-Mahowald, Burklund-Xu:

h_6^2 survives $\Leftrightarrow$ $\lambda \eta \theta_5^2 = 0$ in $\pi_{*,*} S$

(h_6^2 survives to E_{r+3}-page $\Leftrightarrow$ $\eta \theta_5^2 = 0$ in $\pi_{*,*} S/\lambda^r$)

• no η-ext'n from $(124,10)$ to $(125,14)$

Future Goals:

- $\theta_5^2 \overset{?}{=} 0$
- $2\theta_6 \overset{?}{=} 0$
- explicit differentials on h_7^2