Machine Proofs for Adams Differentials and Extension Problems Among CW Spectra

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Computer programs

I have written three major computer programs that are useful in the project of solving the last Kervaire invariant one problem.

- `./Adams` — A program that computes E_2 pages of classical Adams spectral sequences, maps between these E_2 pages and the d_2 Adams differentials.
- `./ss` — A program that computes the Adams differentials and f-extensions for maps f. It can produce human readable proofs.
- `plot.html/main.js` — A webpage program that plots the spectral sequences and the extensions.

`https://waynelin92.github.io/ss/kervaire.html`
`https://zenodo.org/records/14276440`

The program `./Adams`

In algebraic topology, we often need to compute certain Ext groups. For example, the E_2 page of the Adams spectral sequence of S^0 at prime 2 is

$$\mathrm{Ext}_A^{*,*}(\mathbb{F}_2, \mathbb{F}_2).$$

Therefore it is beneficial if we have efficient algorithms to compute these groups and also various structures on them.

The program `./Adams`

In the area of computational commutative algebra which has many applications in algebraic geometry, we often consider a commutative algebra of the following form

$$R = k[x_1, \ldots, x_n]/(f_1, \ldots, f_m).$$

If we are given a concrete R homomorphism between finite dimensional free R-modules

$$f : R^n \to R^m,$$

we can use algorithms in computational commutative algebra to compute the kernel of the map and obtain an exact sequence

$$R^\ell \xrightarrow{k} R^n \xrightarrow{f} R^m.$$

The program `./Adams`

$$\cdots \to R^\ell \xrightarrow{k} R^n \xrightarrow{f} R^m \to \operatorname{coker}(f).$$

If we repeat this process we can obtain a (minimal) free resolution of $M = \operatorname{coker}(f)$.
The foundation of these algorithms are the notation of Gröbner basis and the Buchberger algorithm.

The program `./Adams`

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However, many important algebras in algebraic topology such as the Steenrod algebra are not commutative. One would like to generalize the theory of Gröbner basis and also algorithms based on it so that we can apply the generalized algorithms to noncommutative algebras such as the Steenrod algebra.

The program `./Adams`

Theorem (L.)

There is a noncommutative generalization of Gröbner basis and associated algorithms which apply to the Steenrod algebra.

The program `./Adams`

Definition

An algebra A over k is called a *(decreasingly-)finitely-filtered-commutative algebra* if it satisfies the following conditions.

1. The algebra A is equipped with a filtration of ideals
$$A = F_0A \supset F_1A \supset F_2A \supset \cdots$$
such that
$$F_pA \cdot F_qA \subset F_{p+q}A$$
and $F_{p_{\mathrm{nul}}}A = 0$ for some number $p_{\mathrm{nul}} > 0$ (in each degree of A if A is graded).
2. The associated graded algebra $\mathrm{gr}(A)$ defined by
$$\mathrm{gr}_p(A) = F_pA/F_{p+1}A$$
is commutative.

The program `./Adams`

The noncommutative Gröbner basis theory we define applies to finitely-filtered-commutative algebras including primitively generated algebras such as the Steenrod algebra, the $\mathbb{C}$-motivic and $\mathbb{R}$-motivic Steenrod algebra.

We can use the new theory to implement the program `./Adams` and obtain the following.

The program `./Adams`

The noncommutative Gröbner basis theory we define applies to finitely-filtered-commutative algebras including primitively generated algebras such as the Steenrod algebra, the $\mathbb{C}$-motivic and $\mathbb{R}$-motivic Steenrod algebra.

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Theorem (L.)

The bigraded algebra

$$\bigoplus_{t\leq 261} \mathrm{Ext}_A^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$$

in internal degrees up to 261 is an algebra with 2914 indecomposables, 23822 basis elements and 227498 indecomposable relations. The complete lists are in given in `https://doi.org/10.5281/zenodo.7786289`.

The program `./Adams`

1. In 2006, Baues showed that we can encode the secondary Steenrod algebra in a differential graded algebra $A^{(2)}$ over $\mathbb{Z}/p^2$ which is explicitly calculated. Baues and Jibladze used this to make an algorithm which computes the Adams d_2 differentials of S^0 up to stem 40.
2. In 2012, Nassau discovered a smaller and simpler presentation of the secondary Steenrod algebra $A^{(2)}$. This can speed up the calculation of Adams d_2 differentials by a lot.
3. In 2022, Chua generalized and implemented the algorithm using $H\mathbb{F}_2$-based synthetic spectra to compute the synthetic homotopy groups $\pi_{*,*}(\nu S^0/\lambda^2)$, which can be roughly thought as the Adams E_3 page of S^0 at the prime 2 including the hidden extensions on the E_3 page which can jump filtration by one.

The program ./ss organizes the spectral sequence by the following.

$$B_2 \subseteq B_3 \subseteq \cdots \subseteq Z_3 \subseteq Z_2 \subseteq E_2.$$

11	$h_0^2 x_{123,9}$	d_2^{-1}	$x_{124,9}$
	$h_5 x_{92,10}$	d_5^{-1}	$x_{124,6}$
	$x_{123,11,2} + x_{123,11} + h_0 h_6 [B_4]$	d_7	$h_1 x_{121,17}$
	$x_{123,11} + h_0 h_6 [B_4]$	d_3	$h_0 x_{122,13}$
	$h_0 h_6 [B_4]$	d_2	$h_0^2 h_6 M d_0$
10	$h_0 x_{123,9}$	d_2^{-1}	$x_{124,8}$
	$x_{123,10} + h_6 [B_4]$	d_3^{-1}	$x_{124,7}$
	$h_6 [B_4]$	d_2	$h_0 h_6 M d_0$
9	$x_{123,9} + h_0 x_{123,8}$	d_{12}	?
	$h_0 x_{123,8}$	d_3	$h_0 x_{122,11} + h_0 h_6 M d_0$
8	$x_{123,8}$	d_3	$x_{122,11} + h_6 M d_0$
$E_2^{*,*}(S^0)$ stem=123			

The program ./ss

The program `./ss` organizes the spectral sequence by the following.

$$B_2 \subseteq B_3 \subseteq \cdots \subseteq Z_3 \subseteq Z_2 \subseteq E_2.$$

Consider

$$d_r : Z_{r-1}/Z_r \xrightarrow{\cong} B_r/B_{r-1}$$

$$d_r^{-1} : B_r/B_{r-1} \xrightarrow{\cong} Z_{r-1}/Z_r$$

We often write differentials in terms of elements in $E_2^{*,*}$. The value of $d_r^{\pm 1}$ is a coset.

- If we have $d_r(x) = y_1$ and $d_r(x) = y_2$ for two different reasons, then we have $y_1 - y_2 \in B_{r-1}$.
- If we have $d_r^{-1}(x) = z_1$ and $d_r^{-1}(x) = z_2$ for two different reasons, then we have $z_1 - z_2 \in Z_r$.

The program `./ss`

The first method to determine $d_r x$.

1. List all possible d_r targets of x.
2. Make assumptions $d_r x = y$.
3. Propagate.
4. rule out enough possibilities by contradictions so that we can determine $d_r x$.

Ways to propagate:

1. Leibniz rule.
2. Naturality.
3. Generalized Leibniz rule.
4. Many other structures and properties.
5. The first method itself.

The program ./ss

Pseudo example on how we determine $d_r x$.

- Assume $d_r(x) = y_1$. Propagate. Contradiction.
- Assume $d_r(x) = y_2$. Propagate. Contradiction.
- Assume $d_r(x) = y_1 + y_2$. The answer.
- Assume $d_r(x) = 0$. Propagate. Contradiction.

The program ./ss

Pseudo example on how we determine $d_r x$.

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- Assume $d_r(x) = y_2$. Propagate. Contradiction.
- Assume $d_r(x) = y_1 + y_2$. The answer.
- Assume $d_r(x) = 0$
  - Assume $d_{r+1}(x) = z_1$. Propagate. Contradiction.
  - Assume $d_{r+1}(x) = z_2$. Propagate. Contradiction.
  - Assume $d_{r+1}(x) = z_1 + z_2$. Propagate. Contradiction.
  - Assume $d_{r+1}(x) = 0$. Propagate. Contradiction.

The program `./ss`

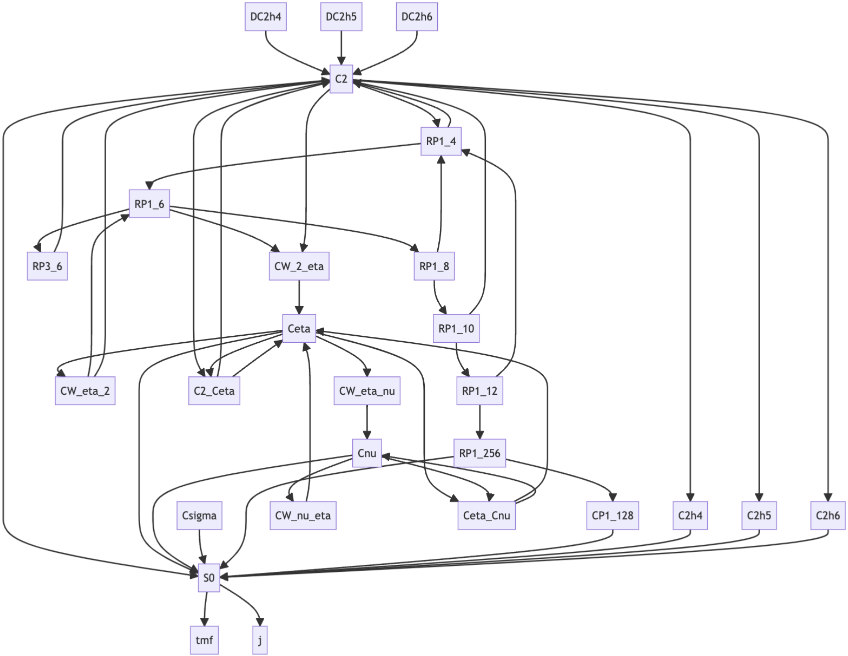
Pseudo example on how we determine $d_r x$. Assume we have a map $f: Z \to X$ and $f(z) = x$. and $f_{E_r}^{-1}(y_4) = \{w_1, w_2\}$.

- Assume $d_r(x) = y_1$. Propagate. Contradiction.
- Assume $d_r(x) = y_2$. Propagate. Contradiction.
- Assume $d_r(x) = y_3$. The answer.
- Assume $d_r(x) = y_4$
    - Assume $d_r(z) = w_1$. Propagate. Contradiction.
    - Assume $d_r(z) = w_2$. Propagate. Contradiction.

The program ./ss

Pseudo example on how we determine $d_r z$. Assume we have a map $f : X \to Z$ and $f(x) = z$.

- Assume $d_r(x) = y_1$. Propagate. Contradiction.
- Assume $d_r(x) = y_2$. Propagate. Contradiction.
- Assume $d_r(x) = y_3$. Also get $d_r(z) = f(y_3) = w$ by naturality.
- Assume $d_r(x) = y_4$. Also Get $d_r(z) = f(y_3) = w$ by naturality.


DC2h4
DC2h5
DC2h6
C2
RP1_4
RP1_6
RP3_6
CW_2_eta
RP1_8
Ceta
RP1_10
CW_eta_2
C2_Ceta
CW_eta_nu
RP1_12
Cnu
RP1_256
Csigma
CW_nu_eta
Ceta_Cnu
CP1_128
C2h4
C2h5
C2h6
S0
tmf
j

Extension spectral sequence

The second method.
Consider a cofiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

We have a long exact sequence of homotopy groups

$$\cdots \to \pi_n X \to \pi_n Y \to \pi_n Z \to \pi_{n-1} X \to \cdots$$

We can consider this as a chain complex over $\pi_* S^0$ filtered by Adams filtration. Thus we get an extensions spectral sequence

$$E_0 \cong E_\infty(X) \oplus E_\infty(Y) \oplus E_\infty(Z) \Longrightarrow 0$$

We call the differentials here extensions. If one does not come from the one induced by E_2-page maps, then it represents a hidden extension.

The program ./ss

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

The program ./ss organizes the spectral sequence by the following.

$$B_2 \subseteq B_3 \subseteq \cdots \subseteq \mathbf{B}_\infty \subseteq \mathbf{Z}_\infty \subseteq \cdots \subseteq Z_3 \subseteq Z_2 \subseteq E_2(Y).$$

$$\mathbf{B}_\infty \subseteq B_0^f \subseteq B_1^f \subseteq \cdots \subseteq Z_1^g \subseteq Z_0^g \subseteq \mathbf{Z}_\infty.$$

$$d_r^f : Z_{r-1}^f(X)/Z_r^f(X) \xrightarrow{\cong} B_r^f(Y)/B_{r-1}^f(Y)$$

$$d_0^f : \mathbf{Z}_\infty(X)/Z_0^f(X) \xrightarrow{\cong} B_0^f(Y)/\mathbf{B}_\infty(Y)$$

We often write differentials in terms of elements in $E_2^{*,*}$. The value of $d_r^{\pm 1}$ is a coset.

- When $r > 0$, if we have $d_r^f(x) = y_1$ and $d_r^f(x) = y_2$ for two different reasons, then we have $y_1 - y_2 \in B_{r-1}^f$.
- If we have $d_0(x) = y_1$ and $d_0(x) = y_2$ for two different reasons, then we have $y_1 - y_2 \in \mathbf{B}_\infty$.

The program `./ss`

The program `./ss` will support homotopy data and its interactions with Adams differentials and extensions. The feature is not complete yet.

Live demo of the program `./Adams`

Ring spectra R.

- `./Adams res R 100` — Compute the minimal resolution of $H^*(R)$ over the Steenrod algebra up to internal degree 100.
- `./Adams prod R 100` — Compute the algebraic structure of $\mathrm{Ext}(R)$.
- `./Adams export R 100` — Export the algebraic structure of $\mathrm{Ext}(R)$ to a file named by `R_AdamsSS.db`.
- `./Adams d2 R 100` — Compute the d_2 differentials.
- `./Adams export_d2 R` — Export the d_2 information to the file `R_AdamsSS.db`.

Live demo of the program `./Adams`

Module X over R.

- `./Adams res X 100` — Compute the minimal resolution of $H^*(R)$ over the Steenrod algebra up to internal degree 100.
- `./Adams prod_mod X R 100` — Compute the module structure of $\mathrm{Ext}(X)$ over $\mathrm{Ext}(R)$.
- `./Adams export_mod X R 100` — Export the module structure of $\mathrm{Ext}(X)$ over $\mathrm{Ext}(R)$ to a file named by `X_AdamsSS.db`.
- `./Adams d2 X 100` — Compute the d_2 differentials.
- `./Adams export_d2 X` — Export the d_2 information to the file `X_AdamsSS.db`.

Live demo of the program `./Adams`

Map $f : X \to Y$.

- `./Adams map_res X Y 100` — Compute the chain map between resolutions induced by $X \to Y$ up to internal degree 100.
- `./Adams export_map X Y 100` — Export the map between E_2 pages from the chain map.

Live demo of the program `./ss`

- `./ss reset demo` — Initialize a category in a folder named by `demo`.
- `./ss deduce auto demo` — Deduce Adams differentials and extensions in the category `demo`.
- `./ss deduce auto demo flags=zero` — the flag `zero` will push the program further in finding contradictions.
- `./ss plot_ss demo` — Plot the spectral sequences.
- `./ss plot_cofseq demo` — Plot the extensions for maps in the cofiber sequences.
- `./ss add_diff S0 15 2 3 0 0 demo` — Manually add a differential `S0 (15,2) d_3[0]=[0]` to `demo`.

Future programming work

- `./ss` computes homotopy groups $\Longrightarrow$ New CW spectra $\Longrightarrow$ `./Adams` computes the new E_2 data $\Longrightarrow$ `./ss` computes more homotopy groups.
- Propagate information using the Leibniz rule from

$$(DX \wedge Y) \wedge (DY \wedge Z) \to DX \wedge Z.$$

- Compute actual homotopy data instead of associated graded ones.
- $\mathbb{C}$-motivic and $\mathbb{R}$-motivic data.
- Odd prime p.
- Slice spectral sequences.

Machine proofs

Our proof for the Last Kervaire Invariant One Problem depends on the machine calculation of some Adams differentials and extensions. You can find most of the machine outcome on the webpage `https://waynelin92.github.io/ss/kervaire.html`. The program starts with the following data.

- The Adams E_2 pages of a collection of CW spectra.
- Maps between these E_2 pages.
- Some d_2 differentials for some CW spectra.
- Three manually added differentials: $d_5 h_0^{24} h_6 = h_0^2 P^6 d_0$, $d_6 h_0^{55} h_7 = h_0^2 x_{126,60}$ in the Adams spectral sequence of S^0 by the image of J knowledge and $d_3 v_2^{16} = \beta^5 g$ in the Adams spectral sequence of *tmf* by power operations (Bruner-Rognes).

The current public version of the plot are only based on the above data. We also have a nonpublic version based on differentials known in the literature.

Machine proofs

We will demonstrate the machine proofs for the following statements. The reader find the proofs in the file `proofs.db` or `proofs.csv` on Zenodo.

1. For $j[0] \in E_2^{10,25+10}(C2)$, we have $d_3(j[0]) = h_1 P d_0(h_1[1])$.
2. For $h_0 h_4 \in E_2^{2,15+2}(S^0)$, we have $d_3(h_0 h_4) = h_0 d_0$.
3. For $\Delta h_2^2 \in E_2^{6,30+6}(S^0)$, we have $d_3(\Delta h_2^2) = h_0^2 k$.
4. For $h_3 h_5 \in E_2^{2,38+2}(S^0)$, we have $d_4(h_3 h_5) = h_0 x$.
5. For $h_5 P e_0 \in E_2^{9,56+9}(S^0)$, we have $d_5(h_5 P e_0) = d_0 \Delta h_0^2 e_0$.
6. For $x_{85,6} + h_0^3 c_3 \in E_2^{6,85+6}(S^0)$, we have $d_9(x_{85,6} + h_0^3 c_3) = x_{84,15,2} + x_{84,15}$.
7. For $h_1 p_1 \in E_2^{5,71+5}(S^0)$, we have $d_5(h_1 p_1) = h_1^2 x_{68,8}$.

The program `./ss`

This is the shape of the proof for determining $d_5(h_1 p_1)$. Consider $r = 5$, $x = h_1 p_1$ and $f : S^1 \to$ CW_2_A_eta.

- Assume $d_r(x) = 0$. This implies that x is a permanent cycle.
  - Assume $d_{r+1}(x) = a$. Propagate. Contradiction.

  Hence $d_{r+1}(x) = 0$. Eventually we get that x is a permanent cycle.
  - Assume $d_s^f(x) = z_1$. Propagate. Contradiction.
  - Assume $d_s^f(x) = z_2$. Propagate. Contradiction.
  - Assume $d_s^f(x) = z_3$. Propagate. Contradiction.
  - Assume $d_s^f(x) = z_4$. Propagate. Contradiction.
- Assume $d_r(x) = y$. The answer.

			8		1			
						4	3	
5								
				7		8		
						1		
	2			3				
6							7	5
		3	4					
			2			6		

Thank You!