

HOMOTOPICAL CATEGORIES

The goal of this talk is to introduce homotopical categories as a framework for talking about abstract homotopy theory, and give a few examples. If you've seen model category theory, this might look familiar. Read [Rie14, Sections 2.1 and 2.2] or [HHR16, Section B.1].

OUTLINE

(1) Homotopical Categories

- (a) Define a homotopical category [Rie14, Definition 2.1.1]: a pair $(\mathcal{C}, \mathcal{W})$ of a category $\mathcal{C}$ and a wide subcategory $\mathcal{W}$ such that $\mathcal{W}$ satisfies the two-out-of-six property. Note that the terminology is to call the morphisms in the class $\mathcal{W}$ the *weak equivalences* of a homotopical category $(\mathcal{C}, \mathcal{W})$.
- (b) Discuss the two-out-of-six property and the two-out-of-three property. The two-out-of-three property is an axiom in model categories; the two-out-of-six property is stronger. You will prove this on your homework.
- (c) Give an example: any category $\mathcal{C}$ becomes a homotopical category if we chose $\mathcal{W}$ to be the class of isomorphisms in $\mathcal{C}$ [Rie14, Example 2.1.4].
- (d) Another example: the category $\mathcal{Top}$ of topological spaces becomes a homotopical category if we choose $\mathcal{W}$ to be the homotopy equivalences. Prove this! [Rie14, Digression 2.1.5].
- (e) Define a weak homotopy equivalence between two topological spaces. The category $\mathcal{Top}$ with weak homotopy equivalences is also a homotopical category, as is the category $\mathcal{Top}_*$ of pointed spaces and with based (weak) homotopy equivalences.
- (f) An algebraic example: let $\mathcal{Ch}(R)$ be the category of unbounded chain complexes of R -modules, for a commutative ring R . Define a quasi-isomorphism between two chain complexes. $\mathcal{Ch}(R)$ becomes a homotopical category with the class $\mathcal{W}$ of quasi-isomorphisms. We might also choose $\mathcal{W}$ to be the chain homotopy equivalences. To draw the analogy with $\mathcal{Top}$, the quasi-isos are the algebraic analogue of weak homotopy equivalences, and the chain homotopy equivalences are the algebraic analogue of homotopy equivalences.
- (g) Variants of the above example include categories of bounded chain complexes, chain complexes of sheaves, etc.
- (h) Define the *homotopy category* $ho(\mathcal{C})$ of a homotopical category $(\mathcal{C}, \mathcal{W})$ and define the localization functor $\mathcal{C} \rightarrow ho(\mathcal{C})$ [Rie14, Definition 2.1.6].
- (i) Describe the universal property of the homotopy category [Rie14, Remark 2.1.11].
- (j) If $\mathcal{W}$ is the class of weak homotopy equivalences in $\mathcal{Top}_*$, then $ho(\mathcal{Top}_*)$ is called the *homotopy category of spaces* or sometimes simply *the homotopy category*.
- (k) If $\mathcal{W}$ is the class of quasi-isomorphisms in $\mathcal{Ch}(R)$, then $\mathcal{D}(R) := ho(\mathcal{Ch}(R))$ is called the *derived category of R* . Variants include bounded derived category, etc.

(2) Derived Functors

- (a) We want to understand functors between homotopical categories. To that end, we define derived functors.
- (b) Define a *homotopical functor* between homotopical categories.
- (c) Give examples: homotopy groups of spaces, homology/cohomology of spaces or of chain complexes. Any exact functor between categories of chain complexes. [Rie14, Examples 2.12, 2.14].

- (d) Give non-examples: [Rie14, Example 2.15] and [Rie14, Example 2.16].
 - (e) Given that there are interesting functors $F: (\mathcal{C}, \mathcal{W}) \rightarrow (\mathcal{D}, \mathcal{U})$ between homotopical categories that are *not* themselves homotopical, we might ask for homotopical approximations to these non-homotopical functors. These approximations are called *derived functors*.
 - (f) Define total left derived functors [Rie14, Definition 2.1.17] of a functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between homotopical categories. By the universal property the localization functor $\gamma: \mathcal{C} \rightarrow \mathrm{ho}(\mathcal{C})$, the total left derived functor $LF: \mathrm{ho}(\mathcal{C}) \rightarrow \mathrm{ho}(\mathcal{D})$ is equivalent to a homotopical functor $\mathcal{C} \rightarrow \mathrm{ho}(\mathcal{D})$.
 - (g) Sometimes, but not always, we can lift this again from a functor $\mathcal{C} \rightarrow \mathrm{ho}(\mathcal{D})$ to a homotopical functor $\mathcal{C} \rightarrow \mathcal{D}$. Define a left derived functor [Rie14, Definition 2.1.19].
 - (h) Give an example of a left derived functor: the left derived functor of $- \otimes_R M$ for a fixed R -module M is $\mathrm{Tor}_R(-, M)$. Make sense of this statement in the context of homotopical categories!
- (3) Kan Extensions
- (a) To talk about derived functors properly, we need to review Kan extensions. We will use this on the problem set this week and next week.
 - (b) Define left and right Kan extensions, following [Rie14, Definition 1.1.1].
 - (c) Give examples: [Rie14, Examples 1.1.6, 1.4.1, 1.4.2]. There are many more examples out there: the slogan is “all concepts are Kan extensions.” Pick a few examples that you find interesting or relevant.
 - (d) Explain what it means for a functor to preserve a Kan extension, following the opening sentences of [Rie14, Section 1.3].
 - (e) Define an *absolute Kan extension*: a Kan extension that is preserved by any functor whatsoever.

REFERENCES

- [HHR16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.