

Reading: Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

- (1) A class of morphisms $\mathcal{W}$ in a category $\mathcal{C}$ satisfies the *two-out-of-three property* if given any two composable morphisms f and g , if any two of f , g , and gf are in $\mathcal{W}$, then so is the third.
 - (a) Prove that the class of weak equivalences $\mathcal{W}$ in a homotopical category $\mathcal{C}$ obeys the two-out-of-three property.
 - (b) Is the two-out-of-three property equivalent to the two-out-of-six property?
- (2) Let $\mathcal{C}$ be any category equipped with a collection of morphisms $\mathcal{W}$. We say that $\mathcal{W}$ is *saturated* if every morphism f in $\mathcal{C}$ which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in $\mathcal{W}$. We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if f becomes an isomorphism in $\mathrm{ho}(\mathcal{C})$, then $f \in \mathcal{W}$).
 - (a) Prove that if $\mathcal{W}$ is saturated, then $\mathcal{W}$ has the two-out-of-six property.
 - (b) Give an example of a homotopical category that is *not* saturated.
 - (c) (Optional) Show that the class of weak equivalences in a model category is saturated.
- (3) An *absolute left (or right) Kan extension* is a Kan extension that is preserved by any functor whatsoever. Let $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ be an adjunction between homotopical categories $\mathcal{C}$ and $\mathcal{D}$. Let $LF: \mathrm{ho}(\mathcal{C}) \rightarrow \mathrm{ho}(\mathcal{D})$ be the total left derived functor for F and $RG: \mathrm{ho}(\mathcal{D}) \rightarrow \mathrm{ho}(\mathcal{C})$ be the total right derived functor for G . Assume that LF and RG are absolute right/left Kan extensions. Prove that LF is left adjoint to RG .
- (4) Let $\mathcal{C}$ be a homotopical category and let $L: \mathcal{C} \rightarrow \mathrm{ho}(\mathcal{C})$ be the localization functor.
 - (a) Let $c \in \mathcal{C}$. Prove that any natural transformation $\mathcal{C}(c, -) \Rightarrow F$ factors through $\mathrm{ho}(\mathcal{C})(c, -)$, where $F: \mathcal{C} \rightarrow \mathbf{Set}$ is a homotopical functor.
 - (b) Let $c \in \mathcal{C}$ be an object such that $\mathcal{C}(c, -)$ is a homotopical functor. Prove that the natural transformation $\mathcal{C}(c, -) \rightarrow \mathrm{ho}(\mathcal{C})(c, -)$ induced by L is a natural bijection.

REFERENCES

- [HHR16] Michael Hill, Michael Hopkins, and Douglas Ravenel. On the nonexistence of elements of Kervaire invariant one. *Ann. of Math. (2)*, 184(1):1–262, 2016.
- [Mac71] Saunders MacLane. *Categories for the working mathematician*, volume Vol. 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1971.
- [Mal07] Georges Maltsiniotis. Le théorème de Quillen, d’adjonction des foncteurs dérivés, revisité. *C. R. Math. Acad. Sci. Paris*, 344(9):549–552, 2007.
- [Qui67] Daniel G. Quillen. *Homotopical algebra*, volume No. 43 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin-New York, 1967.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.