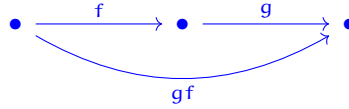


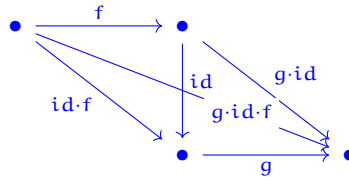
Reading: Read §2.1 and §2.2 in [Rie14] or §B.1 in [HHR16].

- (1) A class of morphisms $\mathcal{W}$ in a category $\mathcal{C}$ satisfies the *two-out-of-three property* if given any two composable morphisms f and g , if any two of f , g , and gf are in $\mathcal{W}$, then so is the third.
- (a) Prove that the class of weak equivalences $\mathcal{W}$ in a homotopical category $\mathcal{C}$ obeys the two-out-of-three property.

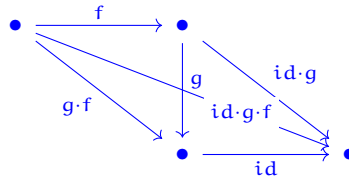
SOLUTION: Let $\mathcal{C}$ be a homotopical category with weak equivalences the subcategory $\mathcal{W}$. We will show that the $\mathcal{W}$ has the two-out-of-three property by using its two-out-of-six property. Consider the morphisms f , g , and gf in the following diagram:



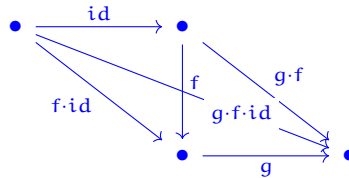
Suppose $f, g \in \mathcal{W}$. Then $gf \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



Suppose $g, gf \in \mathcal{W}$, then $f \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



Finally, suppose $f, gf \in \mathcal{W}$, then $g \in \mathcal{W}$ by the two-out-of-six property applied to the diagram:



- (b) Is the two-out-of-three property equivalent to the two-out-of-six property?

SOLUTION: No. It is weaker. For a counterexample, consider the category:

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$$

Let $\mathcal{W} = \{gf, hg, id_A, id_B, id_C, id_D\}$. $\mathcal{W}$ clearly satisfies the two-out-of-six property. It does not satisfy the two-out-of-three property: gf and hg are in $\mathcal{W}$, while hgf is not.

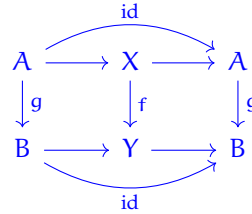
- (2) Let $\mathcal{C}$ be any category equipped with a collection of morphisms $\mathcal{W}$. We say that $\mathcal{W}$ is *saturated* if every morphism f in $\mathcal{C}$ which becomes an isomorphism in $\mathcal{C}[\mathcal{W}^{-1}]$ is in $\mathcal{W}$. We say that a homotopical category is *saturated* if the class of weak equivalences is saturated (i.e. if f becomes an isomorphism in $\text{ho}(\mathcal{C})$, then $f \in \mathcal{W}$).

- (a) Prove that if $\mathcal{W}$ is saturated, then $\mathcal{W}$ has the two-out-of-six property.

SOLUTION: Suppose f, g, h are composable in $\mathcal{C}$ and $fg, gh \in \mathcal{W}$. Then, fg and gh are isomorphisms in $\text{ho } \mathcal{C}$. Isomorphisms satisfy 2-out-of-6, so f, g, h are isomorphisms in $\text{ho } \mathcal{C}$. Thus, $f, g, h \in \mathcal{W}$. This proves the claim. [Rie14, Lemma 2.1.10]

- (b) Give an example of a homotopical category that is *not* saturated.

SOLUTION: Consider the following category, which represents a morphism of retracts.



and let the weak equivalence be given by f and the identity morphisms. This satisfies 2-out-of-6 since the only way to obtain f via a composite of two morphisms is with f and an identity, and the only way to obtain an identity via a composite of two morphisms either uses identities or is one of the horizontal composites, but the maps $X \rightarrow A$ and $Y \rightarrow B$ cannot be applied before another morphism to either give f or an identity.

The weak equivalences are not closed under retract, since g is not a weak equivalence, but isomorphisms are closed under retract, so g must be an isomorphism in the homotopy category. Thus, this construction is a homotopical category which is not saturated.

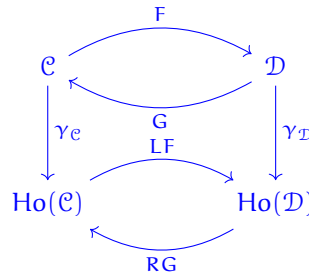
This example is nice because it is clear that it cannot have a model structure, as the weak equivalences in a model category are closed under retract by definition.

- (c) (Optional) Show that the class of weak equivalences in a model category is saturated.

SOLUTION: [Rie14, Remark 2.1.9], which refers to [Qui67, Proposition 5.1]

- (3) An *absolute left (or right) Kan extension* is a Kan extension that is preserved by any functor whatsoever. Let $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ be an adjunction between homotopical categories $\mathcal{C}$ and $\mathcal{D}$. Let $LF: \text{ho}(\mathcal{C}) \rightarrow \text{ho}(\mathcal{D})$ be the total left derived functor for F and $RG: \text{ho}(\mathcal{D}) \rightarrow \text{ho}(\mathcal{C})$ be the total right derived functor for G . Assume that LF and RG are absolute right/left Kan extensions. Prove that LF is left adjoint to RG .

SOLUTION: Consider the diagram:



where F and G are adjoint functors with unit $\eta: \text{id}_C \Rightarrow G \circ F$ and counit $\epsilon: F \circ G \Rightarrow \text{id}_D$, (LF, α) is a total left derived functor where $\alpha: LF \circ \gamma_C \Rightarrow \gamma_D \circ F$, and (RG, β) is a total right derived

functor where $\beta : \gamma_{\mathcal{C}} \circ G \Rightarrow RG \circ \gamma_{\mathcal{D}}$. Our aim is to show that LF and RG are adjoint. We will first produce the unit and counit of a potential adjunction. Since LF is an absolute right Kan extension, $RG \circ LF \cong \text{Lan}_{\lambda}(RG \circ \gamma_{\mathcal{D}} \circ F)$ with natural transformation $RG\alpha$ (the whiskered composite):

$$\begin{array}{ccccccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} & \xrightarrow{\gamma_{\mathcal{D}}} & \text{Ho}(\mathcal{D}) & \xrightarrow{RG} & \text{Ho}(\mathcal{C}) \\
 \downarrow \gamma_{\mathcal{C}} & & & \nearrow LF & & \nearrow \text{Lan}_{\lambda}(RG \circ \gamma_{\mathcal{D}} \circ F) & \\
 \text{Ho}(\mathcal{C}) & & & & & &
 \end{array}$$

We would like to apply the universal property of the pair $(RG \circ LF, RG\alpha)$ to the functor $\text{Id}_{\text{Ho}(\mathcal{C})}$. To do so requires coming up with a natural transformation from $\text{Id}_{\text{Ho}(\mathcal{C})}$ to $RG \circ \gamma_{\mathcal{D}} \circ F$. Such a natural transformation is given by:

$$\beta F \circ \gamma_{\mathcal{C}} \eta : \text{Id}_{\text{Ho}(\mathcal{C})} \circ \gamma_{\mathcal{C}} = \gamma_{\mathcal{C}} \circ \text{Id}_{\mathcal{C}} \Rightarrow \gamma_{\mathcal{C}} \circ G \circ F \Rightarrow RG \circ \gamma_{\mathcal{D}} \circ F$$

Hence, we get a unique natural transformation $\bar{\eta} : \text{Id}_{\text{Ho}(\mathcal{C})} \Rightarrow RG \circ LF$. A similar argument produces a natural transformation $\bar{\epsilon} : RG \circ LF \Rightarrow \text{Id}_{\text{Ho}(\mathcal{D})}$ coming from the existence of the natural transformation $\gamma_{\mathcal{D}} \epsilon \circ \alpha G : LF \circ \gamma_{\mathcal{C}} \circ G \Rightarrow \gamma_{\mathcal{D}} \circ \text{Id}_{\mathcal{D}}$. A convenient way to determine whether $\bar{\eta}$ and $\bar{\epsilon}$ form an adjunction is to check the *triangular identities* (see page 85 of [Mac71]):

$$\begin{aligned}
 \bar{\epsilon} LF \circ LF \bar{\eta} &= \text{Id}_{LF}, \\
 RG \bar{\epsilon} \circ \bar{\eta} RG &= \text{Id}_{RG}
 \end{aligned}$$

We will show the first one as the second follows a similar argument. Since the universal property ensures a unique factorization for derived functors, it suffices to show:

$$\alpha \circ (\bar{\epsilon} LF \circ LF \bar{\eta}) \gamma_{\mathcal{C}} = \alpha$$

$$\begin{aligned}
 \alpha \circ (\bar{\epsilon} LF \circ LF \bar{\eta}) \gamma_{\mathcal{C}} &= \alpha \circ \bar{\epsilon} (LF \circ \gamma_{\mathcal{C}}) \circ (LF) \bar{\eta} \gamma_{\mathcal{C}} \\
 &= (\bar{\epsilon} (\gamma_{\mathcal{D}} \circ F)) \circ ((LF \circ RG) \alpha) \circ (LF) \bar{\eta} \gamma_{\mathcal{C}} \\
 &= \bar{\epsilon} (\gamma_{\mathcal{D}} \circ F) \circ LF ((RG) \alpha \circ \bar{\eta} \gamma_{\mathcal{C}}) \\
 &= \bar{\epsilon} (\gamma_{\mathcal{D}} \circ F) \circ LF (\beta F \circ \gamma_{\mathcal{C}} \eta) \\
 &= \bar{\epsilon} (\gamma_{\mathcal{D}} \circ F) \circ (LF) \beta F \circ (LF \circ \gamma_{\mathcal{C}}) \eta \\
 &= (\bar{\epsilon} \gamma_{\mathcal{D}} \circ (LF) \beta) F \circ (LF \circ \gamma_{\mathcal{C}}) \eta \\
 &= (\gamma_{\mathcal{D}} \epsilon \circ \alpha G) F \circ (LF \circ \gamma_{\mathcal{C}}) \eta \\
 &= \gamma_{\mathcal{D}} \epsilon F \circ \alpha (G \circ F) \circ (LF \circ \gamma_{\mathcal{C}}) \eta \\
 &= \gamma_{\mathcal{D}} \epsilon F \circ (\gamma_{\mathcal{D}} \circ F) \eta \circ \alpha \text{id}_{\mathcal{C}} \\
 &= \gamma_{\mathcal{D}} (\epsilon F \circ F \eta) \circ \alpha \\
 &= \gamma_{\mathcal{D}} \text{id}_F \circ \alpha \\
 &= \text{id}_{\gamma_{\mathcal{D}} \circ F} \circ \alpha \\
 &= \alpha
 \end{aligned}$$

This is from [Rie14, Exercise 2.2.15], which cites [Mal07].

- (4) Let $\mathcal{C}$ be a homotopical category and let $L : \mathcal{C} \rightarrow \text{ho}(\mathcal{C})$ be the localization functor.

- (a) Let $c \in \mathcal{C}$. Prove that any natural transformation $\mathcal{C}(c, -) \Rightarrow F$ factors through $\mathrm{ho}(\mathcal{C})(c, -)$, where $F: \mathcal{C} \rightarrow \mathbf{Set}$ is a homotopical functor.

SOLUTION: This is [HHR16, Proposition B.4]. By the Yoneda lemma,

$$\mathrm{Nat}(\mathcal{C}(c, -), F) \cong F(c)$$

where $x \in F(c)$ corresponds to the natural transformation $\eta(x)$ with components

$$\eta(x)_d: \mathcal{C}(c, d) \rightarrow F(d): f \mapsto fx.$$

Since F is homotopical, and $\mathbf{Set}$ is its own homotopy category, there is a diagram

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathbf{Set} \\ \downarrow L & & \parallel \\ \mathrm{ho} \mathcal{C} & \xrightarrow{\bar{F}} & \mathbf{Set} \end{array}$$

witnessing that F factors through the localization L . Further, $\bar{F}$ is the same as F on objects.

Thus $x \in F(c)$ also corresponds to the natural transformation $\bar{\eta}(x): \mathrm{ho}(\mathcal{C})(c, -) \Rightarrow \bar{F}$ with components

$$\bar{\eta}(x)_d: \mathrm{ho}(\mathcal{C})(c, d) \rightarrow F(d): f \mapsto fx.$$

This witnesses that an arbitrary natural transformation $\eta(x)$ is the composite of $L: \mathcal{C}(c, -) \Rightarrow \mathrm{ho}(\mathcal{C})(c, -)$ and $\bar{\eta}(x) \circ L: \mathrm{ho}(\mathcal{C})(c, -) \Rightarrow \bar{F} \circ L = F$ which proves the desired claim.

- (b) Let $c \in \mathcal{C}$ be an object such that $\mathcal{C}(c, -)$ is a homotopical functor. Prove that the natural transformation $\mathcal{C}(c, -) \rightarrow \mathrm{ho}(\mathcal{C})(c, -)$ induced by L is a natural bijection.

SOLUTION: This is [HHR16, Corollary B.6]. By the previous problem, the identity natural transformation $\mathcal{C}(c, -)$ factors through $L: \mathcal{C}(c, -) \rightarrow \mathrm{ho}(\mathcal{C})(c, -)$, and therefore L is injective.

Since $\mathcal{C}(c, -)$ is homotopical, if $f: d \rightarrow e$ is a weak equivalence, the composition map $f \circ -: \mathcal{C}(c, f): \mathcal{C}(c, d) \rightarrow \mathcal{C}(c, e)$ is an isomorphism. For $f: d \rightarrow c$ a weak equivalence, let $g \in \mathcal{C}(c, d)$ be the preimage of $1 \in \mathcal{C}(c, c)$ along the isomorphism $f \circ -: \mathcal{C}(c, d) \rightarrow \mathcal{C}(c, c)$. By construction, $f \circ g = 1$. This allows every zig-zag in $\mathrm{ho}(\mathcal{C})(c, d)$ to be rewritten in terms of forward facing arrows, demonstrating that $L: \mathcal{C}(c, -) \rightarrow \mathrm{ho}(\mathcal{C})(c, -)$ is surjective.

This completes the proof.

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