

HOMOTOPY LIMITS AND COLIMITS

The goal of this talk is to develop a little bit of the theory of homotopy limits and colimits via derived functors in a homotopical category. If you've seen some model category theory, this might look a little familiar. And if you're curious, the standard model categories reference is [DS95].

OUTLINE

- (1) Left deformations of homotopical categories [Rie14, Section 2.2]
 - (a) introduce left deformations of homotopical categories [Rie14, def 2.2.1] and left deformations of functors [Rie14, def 2.2.4]
 - (b) The goal is to state [Rie14, theorem 2.2.8], but you don't need to prove it if you don't want to give a proof in the talk.
 - (c) You can say as much or as little about model categories as you like here, depending on how much you know and or like model categories.
 - (d) After theorem 2.2.8, you should skip the rest of section 2.2
- (2) Theory of homotopy limits and colimits
 - (a) introduce the idea of homotopy colimits of shape $\mathcal{J}$ as left derived functors of the colimit functor $\text{colim}: \mathcal{C}^{\mathcal{J}} \rightarrow \mathcal{C}$.
 - (b) Explain the difficulty that homotopy colimits are very rarely colimits in the homotopy category: this is explained in [Rie14, Remark 6.3.1]
 - (c) Make explicit what we need to compute a homotopy colimit: a left deformation of the colim functor
 - (d) Maybe say the words "the bar construction gives a left deformation of $\mathcal{C}^{\mathcal{J}}$ " but you really shouldn't define the bar construction in general if you don't want to
- (3) Homotopy colimits in the category of spaces
 - (a) Explain that CW approximation (replacing a space X by a weakly equivalent CW complex X') is a left deformation on the homotopical category of CGWH spaces (this is cofibrant replacement, if you're a model category person). To get a functorial cofibrant replacement, we usually take the geometric realization of the singular simplicial set: $X \mapsto |\text{Sing}(X)|$.
 - (b) Say that you can get a left deformation on the category of diagrams in spaces by the bar construction, and give the bar construction in this context (see [Mal23, 7.3.23])
- (4) Examples
 - (a) Explain explicitly how to construct the homotopy pushout, homotopy cofiber (mapping cylinder), and homotopy sequential colimit (mapping telescope).
 - (b) Make sure to include the example of the unreduced suspension as a homotopy pushout.
 - (c) If you have time, explain the homotopy pullback and homotopy fiber. A good reference for this (with pictures!) is [Mal23, Section 1.5]

REFERENCES

- [DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995.
- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.