

Reading: Read §2.2 in [Rie14] and §1.5 in [Mal23].

- (1) Let $\mathcal{C}$ be a homotopical category, and let $\mathcal{J}$ be any category. The category $\text{Fun}(\mathcal{J}, \mathcal{C})$ of functors from $\mathcal{J}$ to $\mathcal{C}$ becomes a homotopical category with weak equivalences defined object-wise. By choosing a homotopical category $\mathcal{C}$ and a category $\mathcal{J}$, show that the limit functor

$$\lim: \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}, \quad F \mapsto \lim F$$

is *not* a homotopical functor.

- (2) Let $\mathcal{C}$ be a homotopical category. For any discrete (the only morphisms are identities) category $\mathcal{J}$ with finitely many objects, $\text{ho}(\mathcal{C})^{\mathcal{J}}$ is equivalent to $\text{ho}(\mathcal{C}^{\mathcal{J}})$, where $\mathcal{C}^{\mathcal{J}}$ has weak equivalences defined pointwise [Rie14, Remark 6.3.1]. Use this fact to prove that finite products in $\text{ho}(\mathcal{C})$ are homotopy products and finite coproducts in $\text{ho}(\mathcal{C})$ are homotopy coproducts*.
- (3) A *coequalizer* is the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$ in a category.

- (a) Prove that the data of the coequalizer of two parallel morphisms $A \begin{smallmatrix} \xrightarrow{f} \\ \xrightarrow{g} \end{smallmatrix} B$ is equivalent to the data of the pushout of the diagram

$$A \xleftarrow{\nabla} A \amalg A \xrightarrow{(f,g)} B,$$

where $\nabla: A \amalg A \rightarrow A$ is the fold map.

- (b) Prove that the homotopy coequalizer is equivalent to the homotopy pushout of the diagram in part (a).
- (c) Use part (b) to describe the homotopy coequalizer of two maps in the category Top of (unpointed) topological spaces[†]. Draw a picture to illustrate your description.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.

*Note however that homotopy (co)limits are not, in general, (co)limits in the homotopy category (by problem 1).

[†]To be precise, we assume all spaces are compactly generated and weakly Hausdorff. Or equivalently, assume that all spaces are homotopy equivalent to a CW complex.