

Reading: Read §2.2 in [Rie14] and §1.5 in [Mal23].

- (1) Let $\mathcal{C}$ be a homotopical category, and let $\mathcal{J}$ be any category. The category $\text{Fun}(\mathcal{J}, \mathcal{C})$ of functors from $\mathcal{J}$ to $\mathcal{C}$ becomes a homotopical category with weak equivalences defined object-wise. By choosing a homotopical category $\mathcal{C}$ and a category $\mathcal{J}$, show that the limit functor

$$\lim: \text{Fun}(\mathcal{J}, \mathcal{C}) \rightarrow \mathcal{C}, \quad F \mapsto \lim F$$

is *not* a homotopical functor.

SOLUTION: Plenty of examples. Homotopy pullbacks are not pullbacks. $S^1 \times_* S^1$ is a torus, but $S^1 \times_{D^2} S^1$ is homeomorphic to S^1 .

- (2) Let $\mathcal{C}$ be a homotopical category. For any discrete (the only morphisms are identities) category $\mathcal{J}$ with finitely many objects, $\text{ho}(\mathcal{C})^{\mathcal{J}}$ is equivalent to $\text{ho}(\mathcal{C}^{\mathcal{J}})$, where $\mathcal{C}^{\mathcal{J}}$ has weak equivalences defined pointwise [Rie14, Remark 6.3.1]. Use this fact to prove that finite products in $\text{ho}(\mathcal{C})$ are homotopy products and finite coproducts in $\text{ho}(\mathcal{C})$ are homotopy coproducts*.

SOLUTION:

Assume that $\mathcal{C}$ has homotopy products and $\text{ho } \mathcal{C}$ has products. Then, the homotopy product functor $\text{ho}(\mathcal{C}^{\mathcal{J}}) \rightarrow \text{ho } \mathcal{C}$ is the right derived functor of the product $\mathcal{C}^{\mathcal{J}} \rightarrow \mathcal{C}$. By the previous problem set, this is the adjoint of the left derived functor of the diagonal, $\mathcal{C} \rightarrow \mathcal{C}^{\mathcal{J}}$. Since the diagonal is already homotopical, its left derived functor is again the diagonal $\text{ho}(\mathcal{C})^{\mathcal{J}} \rightarrow \text{ho } \mathcal{C}$ after identifying $\text{ho}(\mathcal{C}^{\mathcal{J}})$ and $(\text{ho } \mathcal{C})^{\mathcal{J}}$. But the adjoint to this diagonal is exactly the homotopy product in $\text{ho } \mathcal{C}$. See also [Rie14, Remark 6.3.1 and footnote 3 therein].

- (3) A *coequalizer* is the colimit of a diagram of shape $\bullet \rightrightarrows \bullet$ in a category.

- (a) Prove that the data of the coequalizer of two parallel morphisms $A \xrightarrow{f} B$ is equivalent to the data of the pushout of the diagram

$$A \xleftarrow{\nabla} A \amalg A \xrightarrow{(f,g)} B,$$

where $\nabla: A \amalg A \rightarrow A$ is the fold map.

SOLUTION: We will show that each object has the universal property of the other. First, assume that P is a pushout in the diagram below

$$\begin{array}{ccc} A \amalg A & \xrightarrow{(f,g)} & B \\ \downarrow \nabla & & \downarrow p \\ A & \xrightarrow{q} & P \end{array}$$

First, claim that p equalizes f and g . To see this, consider the composite $p(f, g)i_1$, where $i_1: A \rightarrow A \amalg A$ is the left injection into the coproduct. We have $(f, g)i_1 = f$, and $\nabla i_1 = \text{id}_A$. Then:

$$pf = p(f, g)i_1 = q\nabla i_1 = q$$

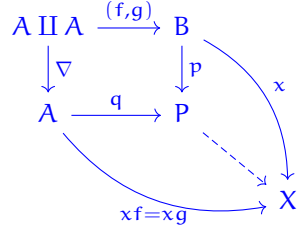
Similarly, $(f, g)i_2 = g$ and $\nabla i_2 = \text{id}_A$, so:

$$pg = p(f, g)i_2 = q\nabla i_2 = q$$

*Note however that homotopy (co)limits are not, in general, (co)limits in the homotopy category (by problem 1).

So $pf = pg$. So p equalizes f and g . It remains to be seen that it is the universal morphism that does so.

Let $x: B \rightarrow X$ be any morphism such that $xf = xg$. Then $x(f, g)i_1 = xf = xg = x(f, g)i_2$, which is the same as $x(f, g) = xf\nabla = xg\nabla$. Then we can draw a commuting diagram



and fill it in with the dashed arrow, which shows that there is a unique morphism $P \rightarrow X$ exhibiting P as the coequalizer of f and g .

The converse is similar.

- (b) Prove that the homotopy coequalizer is equivalent to the homotopy pushout of the diagram in part (a).

SOLUTION: Let $\mathcal{J} = (\bullet \rightrightarrows \bullet)$ be the free parallel arrow category and let $\mathcal{S} = (\bullet \leftarrow \bullet \rightarrow \bullet)$ be the free span category. A coequalizer is the colimit of a diagram of shape $\mathcal{J}$ in a category, and a pushout is the colimit of a diagram of shape $\mathcal{S}$. There are functors

$$\text{colim}_{\mathcal{J}}: \mathcal{J}\text{op}^{\mathcal{J}} \rightarrow \mathcal{J}\text{op},$$

$$\text{colim}_{\mathcal{S}}: \mathcal{J}\text{op}^{\mathcal{S}} \rightarrow \mathcal{J}\text{op},$$

where $\mathcal{J}\text{op}^{\mathcal{C}} := \text{Fun}(\mathcal{C}, \mathcal{J}\text{op})$. There is also a functor

$$F: \mathcal{J}\text{op}^{\mathcal{J}} \rightarrow \mathcal{J}\text{op}^{\mathcal{S}}$$

that sends a pair of parallel arrows $A \xrightarrow[f]{g} B$ to the diagram $A \xleftarrow{\nabla} A \amalg A \xrightarrow{(f,g)} B$.

Part (a) shows that $F \circ \text{colim}_{\mathcal{J}}$ is naturally isomorphic to $\text{colim}_{\mathcal{S}}$ as functors $\mathcal{J}\text{op}^{\mathcal{S}} \rightarrow \mathcal{J}\text{op}$. Let $Q: \mathcal{J}\text{op}^{\mathcal{S}} \rightarrow \mathcal{J}\text{op}^{\mathcal{S}}$ be a left deformation for $\text{colim}_{\mathcal{S}}$. Then this is also a left deformation for $F \circ \text{colim}_{\mathcal{J}}$, since the two functors are naturally isomorphic. Hence, the homotopy coequalizer of f and g is the homotopy pushout of ∇ and (f, g) .

- (c) Use part (b) to describe the homotopy coequalizer of two maps in the category $\mathcal{J}\text{op}$ of (unpointed) topological spaces[†]. Draw a picture to illustrate your description.

SOLUTION: The homotopy coequalizer of two parallel maps $f, g: X \rightarrow Y$ is the mapping torus

$$X \times [0, 1] \sqcup Y / \sim$$

where $\sim$ identifies

$$(x, 0) \sim f(x) \text{ and } (x, 1) \sim g(x).$$

[†]To be precise, we assume all spaces are compactly generated and weakly Hausdorff. Or equivalently, assume that all spaces are homotopy equivalent to a CW complex.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.