

STABLE PHENOMENA IN SPACES TALK OUTLINE

The goal of this talk is to describe spaces as a homotopical category, and to introduce many of the ideas that originally led to spectra. A good reference for this is [Sto22, Chapter 2].

OUTLINE

(1) What do we mean by spaces?

- (a) define categories $\mathcal{Top}$ and $\mathcal{Top}_*$ of CGWH spaces and pointed CGWH spaces
- (b) explain that $\mathcal{Top}$ is symmetric monoidal for cartesian product, $\mathcal{Top}_*$ is closed symmetric monoidal for smash
- (c) we make these into homotopical categories by taking the weak equivalences to be the homotopy equivalences [Rie14, Digression 2.1.5]
- (d) define the function spaces $F(X, Y)$ in $\mathcal{Top}_*$ and the adjunction $X \wedge (-) \dashv F(X, -)$, and note that this adjunction extends to one on function spaces $F(X \wedge Y, Z) \cong F(Y, F(X, Z))$
- (e) note that $\pi_0 F(X, Y) = [X, Y]$
- (f) explain that if we didn't use CGWH spaces, this symmetric monoidal stuff would fail (see Riehl Lemma 6.1.3)

(2) Suspension and loopspaces

- (a) define the suspension $\Sigma X = S^1 \wedge X$ and the loopspace $\Omega X = F(S^1, X)$. This is a special case of the adjunction above
- (b) give the example that $\Sigma S^n \cong S^{n+1}$
- (c) ΣX is the homotopy pushout of $* \leftarrow X \rightarrow *$ and ΩX is the homotopy pullback of the same
- (d) note that $\Sigma \dashv \Omega$, and that this adjunction works both on the level of function spaces $F(-, -)$ and homotopy classes of maps $[-, -]$
- (e) $\pi_n F(X, Y) = [\Sigma^n X, Y]$
- (f) Define H-group. ΩX is an H-group. The consequence of this is that $[Y, \Omega X]$ becomes a group
- (g) mention that ΣX is a co-H-group

(3) Freudenthal Suspension Theorem

- (a) recall the definition of n -connected space
- (b) define the map $\pi_k X \rightarrow \pi_{k+1} \Sigma X$
- (c) state the Freudenthal suspension theorem (but do not prove it) (pick your favorite formulation)
- (d) state [Sto22, corollary 2.23]
- (e) define stable homotopy groups $\pi_*^s(X)$
- (f) (if you have time) define the space QX and show that it's homotopy groups are the same as the homotopy groups of X

(4) Blakers–Massey theorem

- (a) state the Blakers–Massey theorem in the form of a homotopy pullback/pushout square

- (b) mention that this is homotopy excision (you don't have to prove it)
 - (c) state [Sto22, Corollary 2.29] and prove it from the theorem
- (5) Optional Extra Topics, if you have time:
- (a) Hurewicz as a corollary of the above Freudenthal or Blakers–Massey
 - (b) Stable classes of maps between spaces $\{X, Y\}$
 - (c) Localization of spaces at the stable equivalences / Spanier–Whitehead Category
 - (d) Puppe sequences, cofiber seq gives LES in homology, fiber seq gives LES in homotopy

REFERENCES

- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.
- [Sto22] Bruno Stonek. Introduction to stable homotopy theory. <https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf>, July 2022.