

Reading: [Sto22, Chapter 2].

- (1) A theorem of Serre shows that $\pi_i(S^n)$ for $i > 2$ is a finite abelian group, except for two classes of exceptions: $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus M$, where M is a torsion $\mathbb{Z}$ -module. Use this to prove that the stable homotopy groups $\pi_i^s(S^0)$ are finite abelian for $i > 0$.
- (2) Let $\mathcal{C}$ be a category with pushouts, and consider the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z, \end{array}$$

where the left square is a pushout.

- (a) *The Pasting Lemma:* Prove that the right square is a pushout square if and only if the outer rectangle is a pushout square.
- (b) Prove that the same is true if the squares are homotopy pushouts instead of pushouts.

Recall that a pointed space X is *well-based* if the inclusion of the basepoint is a cofibration.*

- (c) Let $f: X \rightarrow Y$ be a pointed map of well-based spaces, and let $\text{cof}(f)$ be the homotopy cofiber of f . Prove that the homotopy cofiber of $Y \rightarrow \text{cof}(f)$ is homotopy equivalent to ΣX .
 - (d) Prove the dual statement: if $\text{fib}(f)$ is the homotopy fiber of f , then the homotopy fiber of $\text{fib}(f) \rightarrow X$ is homotopy equivalent to ΩY .
- (3) Let $f: X \rightarrow Y$ be a map between simply connected spaces such that $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for $i \leq n$. We will show that f is an n -connected map.
 - (a) Let C be the homotopy cofiber of f , and let F be the homotopy fiber of $Y \rightarrow C$. Use the Hurewicz theorem to show that C is n -connected and $F \rightarrow Y$ is an n -connected map.
 - (b) Use the Blakers–Massey theorem to show that $X \rightarrow F$ is at least 2-connected.
 - (c) Show that f is at least 2-connected. Iterate your argument from part (b) to show that f is n -connected.
 - (4) Let $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ be a sequence of spaces. Prove that $\Omega \text{hocolim}_i X_i \simeq \text{hocolim}_i \Omega X_i$. Use this to show that homotopy groups commute with sequential homotopy colimits.

REFERENCES

- [Mat13] MathOverflow. Must a weak homotopy equivalence induce an isomorphism between stable homotopy groups? <https://mathoverflow.net/questions/148963/must-a-weak-homotopy-equivalence-induce-an-isomorphism-between-stable-homotopy-g>, 2013.
- [Sto22] Bruno Stonek. Introduction to stable homotopy theory. <https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf>, July 2022.

If spaces are not well-based, then $\Sigma X := X \wedge S^1$ is not always the same as the homotopy pushout of $ \leftarrow X \rightarrow *$. For these to be the same, we must assume well-based. See Tyler Lawson's answer at [Mat13] for a counterexample.