

Reading: [Sto22, Chapter 2].

- (1) A theorem of Serre shows that $\pi_i(S^n)$ for $i > 2$ is a finite abelian group, except for two classes of exceptions: $\pi_n(S^n) \cong \mathbb{Z}$ and $\pi_{4j-1}(S^{2j}) \cong \mathbb{Z} \oplus M$, where M is a torsion $\mathbb{Z}$ -module. Use this to prove that the stable homotopy groups $\pi_i^s(S^0)$ are finite abelian for $i > 0$.

SOLUTION: Recall that the stable homotopy groups of S^0 are defined as follows:

$$\pi_i^s(S^0) = \operatorname{colim}_{n \rightarrow \infty} \pi_{i+n}(\Sigma^n S^0) = \operatorname{colim}_{n \rightarrow \infty} \pi_{i+n}(S^n)$$

Since $i > 0$, the fact that $\pi_n(S^n) \cong \mathbb{Z}$ is not relevant since it only impacts $\pi_0^s(S^0)$. Thus, we need only show that the colimit of each sequence containing $\pi_{4j-1}(S^{2j})$ is finite abelian. For $i > 0$, every sequence in the colimit $\pi_i^s(S^0)$ will eventually stabilize by the Freudenthal Suspension Theorem. Suppose this stabilization occurs at $\pi_{4j-1}(S^{2j})$. This would imply:

$$\pi_{4j-1}(S^{2j}) \cong \pi_{4j}(S^{2j+1})$$

But $\pi_{4j}(S^{2j+1})$ is finite abelian by Serre's Theorem, so stabilization cannot occur at $\pi_{4j-1}(S^{2j})$. Hence, $\pi_i^s(S^0)$ must be finite abelian for $i > 0$.

- (2) Let $\mathcal{C}$ be a category with pushouts, and consider the following commutative diagram in $\mathcal{C}$

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ X & \longrightarrow & Y & \longrightarrow & Z, \end{array}$$

where the left square is a pushout.

- (a) *The Pasting Lemma*: Prove that the right square is a pushout square if and only if the outer rectangle is a pushout square.

SOLUTION: See [?] for some context, and [?] for a proof of the dual statement.

- (b) Prove that the same is true if the squares are homotopy pushouts instead of pushouts. SOLUTION: Recall that homotopy pushouts are computed by cofibrant replacement and literal pushout. Cofibrantly replace the top left cospan $X \leftarrow A \rightarrow B$ to compute the homotopy pushout by literal pushout. Then, cofibrantly replace the entire square, along with $B \rightarrow C$. The square is still a pushout after cofibrant replacement and now the pushout of $Y \leftarrow B \rightarrow C$ can be computed by literal pushout. Now the literal pasting lemma yields that the entire diagram is a literal pushout, which proves the homotopy pasting lemma.

Recall that a pointed space X is *well-based* if the inclusion of the basepoint is a cofibration.*

This problem is [?, Exercise 16 in Chapter 1]. You can probably also find it (without any category theory) in Hatcher.

- (c) Let $f: X \rightarrow Y$ be a pointed map of well-based spaces, and let $\operatorname{cof}(f)$ be the homotopy cofiber of f . Prove that the homotopy cofiber of $Y \rightarrow \operatorname{cof}(f)$ is homotopy equivalent to ΣX .

If spaces are not well-based, then $\Sigma X := X \wedge S^1$ is not always the same as the homotopy pushout of $ \leftarrow X \rightarrow *$. For these to be the same, we must assume well-based. See Tyler Lawson's answer at [Mat13] for a counterexample.

SOLUTION: Consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & \text{cof}(f) & \longrightarrow & \text{cof}(Y \rightarrow \text{cof}(f)) \end{array}$$

Both squares are individually homotopy pushouts. Thus, by the pasting lemma, the rectangle is a homotopy pushout.

Since the pushout of X mapping to points is the suspension of X , this demonstrates that $\text{cof}(Y \rightarrow \text{cof}(f))$ is homotopy equivalent to ΣX . This is where we use the well-based assumption.

- (d) Prove the dual statement: if $\text{fib}(f)$ is the homotopy fiber of f , then the homotopy fiber of $\text{fib}(f) \rightarrow X$ is homotopy equivalent to ΩY .

SOLUTION: Dually, consider the diagram

$$\begin{array}{ccccc} \text{fib}(\text{fib}(f) \rightarrow X) & \longrightarrow & \text{fib}(f) & \longrightarrow & * \\ \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X & \xrightarrow{f} & Y \end{array}$$

Both squares are homotopy pullbacks, so the rectangle is a homotopy pullback by the dual to the homotopy pasting lemma. Since the pushout of points along Y is the loop-space of Y , this demonstrates that $\text{fib}(\text{fib}(f) \rightarrow X)$ is homotopy equivalent to ΩY .

- (3) Let $f: X \rightarrow Y$ be a map between simply connected spaces such that $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for $i \leq n$. We will show that f is an n -connected map.

- (a) Let C be the homotopy cofiber of f , and let F be the homotopy fiber of $Y \rightarrow C$. Use the Hurewicz theorem to show that C is n -connected and $F \rightarrow Y$ is an n -connected map.

SOLUTION: The cofiber C of f is the homotopy pushout of $* \leftarrow X \xrightarrow{f} Y$, giving a Mayer-Vietoris exact sequence

$$\dots \rightarrow H_i(X) \xrightarrow{f_*} H_i(Y) \rightarrow H_i(C) \xrightarrow{\partial} H_{i-1}(X) \xrightarrow{f_*} H_{i-1}(Y) \rightarrow \dots$$

The map $f_*: H_i(X) \rightarrow H_i(Y)$ is an isomorphism for $i \leq n$. By exactness, $H_i(C)$ is then 0 for $i \leq n$. Furthermore, since X and Y are 1-connected, C is 1-connected as well. Assuming $n > 1$, the Hurewicz Theorem can be used to assert that $\pi_2(C) \cong H_2(C) = 0$. C is then 2-connected and we can iterate the argument to get $\pi_3(C) \cong H_3(C)$. This may proceed until we reach $\pi_n(C) \cong H_n(C) = 0$, and hence C is n -connected. To show that $F \rightarrow Y$ is an n -connected map, consider the long exact sequence in homotopy induced by the fiber sequence $F \rightarrow Y \rightarrow C$:

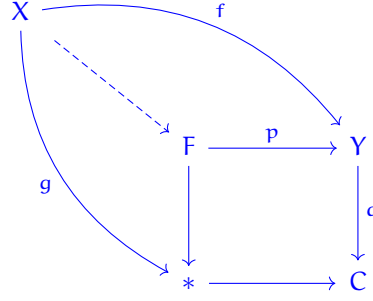
$$\dots \rightarrow \pi_{i+1}(Y) \rightarrow \pi_{i+1}(C) \xrightarrow{\partial} \pi_i(F) \rightarrow \pi_i(Y) \rightarrow \pi_i(C) \rightarrow \dots$$

We have $\pi_i(C) = 0$ for $i \leq n$, so $\pi_i(F) \rightarrow \pi_i(Y)$ is an isomorphism in that range, showing that the map $F \rightarrow Y$ is n -connected.

- (b) Use the Blakers–Massey theorem to show that $X \rightarrow F$ is at least 2-connected.

SOLUTION:

We have the following diagram:



where the outer square is a homotopy pushout and the inner square is a homotopy pullback. Thus, the Blakers-Massey theorem can be applied to the induced map $X \rightarrow F$.

Since X is 1-connected, the map g is at least 2-connected. Furthermore, since Y is also 1-connected, the map f is at least 1-connected. Hence, by the Blakers-Massey theorem, the map $X \rightarrow F$ is at least $(2+1-1=2)$ -connected.

- (c) Show that f is at least 2-connected. Iterate your argument from part (b) to show that f is n -connected.

SOLUTION: We can write f as a composite:

$$f: X \xrightarrow[2]{\text{fib}(g)} Y;$$

the connectivity is drawn under the arrows. f is the composite of a 2-connected map and an n -connected map, so f must be 2-connected.

Now we can repeat the argument of the previous part: by Blakers-Massey, f is 2-connected and $X \rightarrow *$ is 2-connected, so $X \rightarrow \text{fib}(g)$ must be at least $(2+2-1=3)$ -connected. Then f is the composite of a 3-connected map and an n -connected map, so it must be 3-connected. Rinse and repeat to conclude f is n -connected.

- (4) Let $X_0 \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots$ be a sequence of spaces. Prove that $\Omega \text{hocolim}_i X_i \simeq \text{hocolim}_i \Omega X_i$. Use this to show that homotopy groups commute with sequential homotopy colimits.

SOLUTION: This relies on the fact that S^1 is compact, and that we are computing the homotopy colimit. This is a particularly thorny problem, and if you really get into the weeds you'll find yourself questioning what compactness even means. (Compact objects in $\mathcal{Top}$ vs compact spaces – they're different! And S^1 is only the latter.)

Recall that to compute $\text{hocolim}_i X_i$, we cofibrantly replace the sequence X_i with a sequence Y_i and the $\text{hocolim}_i X_i \simeq \text{colim}_i Y_i$. The standard choice is to pick Y_i to be the partial mapping telescopes, and the maps in the sequence of partial mapping telescopes are closed inclusions.

Since $\Omega = F(S^1, -)$ preserves homotopy, the ΩY_i are equivalent to the ΩX_i . Further, the maps between the ΩY_i 's are also closed inclusions (ΩY_i can be identified with the functions into Y_{i+1} that happen to land in Y_i). Therefore, $\text{hocolim}_i \Omega X_i \simeq \text{colim}_i \Omega Y_i$. Thus, it suffices to show that $\Omega \text{colim}_i Y_i \simeq \text{colim}_i \Omega Y_i$.

Let $Y = \text{colim}_i Y_i = \bigcup_i Y_i$. There is a map $\text{colim}_i \Omega Y_i \rightarrow \Omega Y$ induced by the universal property of the colimit. Concretely, it takes a function (or an equivalence class of functions) into a partial mapping telescope and interprets it as a function into the mapping telescope Y .

It is a standard exercise in point set topology that this is a homeomorphism. More generally, if K is compact,

$$\text{colim}_i F(K, Y_i) \rightarrow F(K, Y)$$

is a homeomorphism. We give two arguments:

- (a) To produce the inverse, recognize that the $Y_i \setminus X_i$'s form an open cover of Y , so their preimages cover K . Since K is compact, given any map $K \rightarrow Y$, only finitely many of these preimages are needed to cover K , and therefore K lands in only finitely many of the partial mapping telescopes. If we take Y_i to be the largest such partial mapping telescope, we get a map $K \rightarrow Y_i$. It is straight-forward to show that this is inverse to the canonical map.
- (b) It is not too hard to see that this map is an injection, and that it is a surjection if and only if the natural map $K \rightarrow Y$ factors through Y_i for some i .

So it remains to be seen that $f: K \rightarrow Y$ factors through one of the X_i . We are still assuming that these $Y_i \rightarrow Y_{i+1}$ are closed inclusions. It is now important that these spaces are CGWH. Assume for the sake of contradiction that $f: S^1 \rightarrow Y$ does not factor through any of the Y_i . Then there is $y_0 \in f(K)$ such that $y_0 \notin Y_0$. However, by properties of the colimit, there is $a_0 \in \mathbb{N}$ such that $y_0 \in Y_{a_0}$. Choose $y_1 \in f(S^1)$ such that $y_1 \notin X_{a_0}$. There exists $a_1 \in \mathbb{N}$ such that $y_1 \in Y_{a_1}$. Choose $y_2 \in f(K)$ such that $y_2 \notin Y_{a_1}$. Continue inductively to find a sequence of points $y_0, y_1, y_2, \dots$ such that $y_i \in f(K) \setminus Y_{a_{i-1}}$. Consider the set $Z = \{y_0, y_1, y_2, \dots\} \subseteq f(S^1)$. A subset $K \subseteq Z$ is closed if and only if $K \cap Y_i$ is closed for all i . By construction, $K \cap Y_i$ is finite and therefore closed by weak Hausdorff (in fact, T_1 would be enough). So any subset of Z is closed in Y , including Z itself.

Now consider $f^{-1}(Z) \subseteq S^1$. This is the preimage of a closed set, so closed itself. Therefore, $f^{-1}(Z)$ is compact, as a closed subset of a compact space. But by the above, it is also a discrete topological space since every subset is closed. But any discrete compact space is finite. This is a contradiction, since $f^{-1}(Z)$ is at least countable.

REFERENCES

- [Mat13] MathOverflow. Must a weak homotopy equivalence induce an isomorphism between stable homotopy groups? <https://mathoverflow.net/questions/148963/must-a-weak-homotopy-equivalence-induce-an-isomorphism-between-stable-homotopy-g>, 2013.
- [Sto22] Bruno Stonek. Introduction to stable homotopy theory. <https://bruno.stonek.com/stable-homotopy-2022/stable-online.pdf>, July 2022.