

GENERALIZED COHOMOLOGY TALK OUTLINE

For the Brown Representability Theorem and generalized cohomology theories, see [Hat02, Sections 4.3 and 4.E]. For K-theory, [Hat17, Chapter 2] is a good reference, or [May99, Chapter 24]. I also like [Zak24], because this is where I learned this stuff.

OUTLINE

1. Generalized Cohomology Theories

- (a) Define a generalized cohomology theory following [Hat02, Page 448]: functors $h^n: \text{Top}_* \rightarrow \mathcal{A}b$ with natural isomorphisms $h^n \rightarrow h^{n+1}\Sigma$ satisfying homotopy, exactness, and additivity axioms. (Hatcher uses pointed CW complexes, but we can just use all pointed spaces. The difference between pointed and unpointed stuff is in [Hat02, Theorem 4.59], if you're curious.)
- (b) Construct the cofibration sequence $A \rightarrow X \rightarrow Q \rightarrow \Sigma A \rightarrow \Sigma X \rightarrow \dots$ (where Q is the cofiber of $A \rightarrow X$) using a problem from the second problem set. This is known as a *Puppe Sequence*. Show that this yields a long exact sequence in cohomology for any generalized cohomology theory, using the exactness axiom.
- (c) Give the example of reduced cohomology with coefficients in an abelian group A : $h^i = \tilde{H}^i(-; A)$.
- (d) Another important example of a generalized cohomology theory is K-theory, which we will go into some detail on.

2. K-theory

- (a) Let X be a pointed space. We define $K^0(X)$ to be the free abelian group on isomorphism classes of complex vector bundles over X , modulo the relation that $[E] + [E'] = [E \oplus E']$.
- (b) An example: when $X = *$ is a point, then $K^0(*) \cong \mathbb{Z}$. Vector bundles on a point are just vector spaces, and the only information retained by the isomorphism classes is dimension.
- (c) Another (nontrivial) example: $K^0(S^1) \cong \mathbb{Z}$.
- (d) It's not obvious, but it's a fact that $K^0(X)$ depends on the space X only up to homotopy. If Y is another space such that $Y \simeq X$, then $K^0(Y) \cong K^0(X)$.
- (e) If we want to define a generalized cohomology theory, we have to satisfy the wedge axiom. Note that the wedge axiom implies that $h^i(*) = 0$ for all i , including $i = 0$. See [Hat02, Item (1) at bottom of page 449]. But $K^0(*) = \mathbb{Z}$.
- (f) To fix this, we define *reduced K-theory* as $\tilde{K}^0(X) = \ker(\dim)$, where $\dim: K^0(X) \rightarrow \mathbb{Z}$ is the homomorphism given by $[E] \mapsto \dim(E)$. That is, it takes the dimension of a vector bundle. Then $\tilde{K}^0(*) = 0$.
- (g) This group $\tilde{K}^0(X)$ has an interpretation in terms of vector bundles too: it is the quotient of $K^0(X)$ by the relation that $[E] = [E']$ if there are trivial vector bundles ε^i and ε^j of dimension i and j such that $E \oplus \varepsilon^i \cong E' \oplus \varepsilon^j$, for some $i, j \in \mathbb{N}$. Elements of this group are called *stable equivalence classes of complex vector bundles on X* .
- (h) If we define $\tilde{K}^i(X) = \tilde{K}^0(\Sigma X)$, then this defines a generalized cohomology theory.

3. Generalized Cohomology Theories and Infinite Loopspaces

- (a) An *infinite loopspace* is a space X such that there are spaces X_i for $i \in \mathbb{N}$ and homotopy equivalences $X \simeq \Omega^i X_i$ (we say that $X = X_0$).

- (b) Mention that we also call such a sequence of spaces an Ω -spectrum.
- (c) An example of an infinite loop space is an Eilenberg–MacLane space. Recall the definition of an Eilenberg–MacLane space $K(G, n)$. It is characterized by the property that $\pi_i K(A, n) = 0$ for $i \neq n$ and $\pi_n K(A, n) = A$. This implies that $\Omega K(A, n+1) \simeq K(A, n)$, so each $K(A, n)$ is an infinite loop space.
- (d) Prove that if you have an infinite loop space X , then $h^i = [-, X_n]$ is a generalized cohomology theory. This is [Hat02, Theorem 4.58], but taking the definition of a generalized cohomology theory from [Hat02, Page 448] makes the proof even easier.

4. Brown Representability Theorem

- (a) State the Brown Representability Theorem [Hat02, Theorem 4E.1]. Do not prove it.
- (b) State the theorem that $\tilde{H}^n(X; A) = [X, K(A, n)]$, but do not prove it.
- (c) K-theory is also represented by an infinite loop space. In this case, the infinite loop space is the loop space of BU , the classifying space of the infinite unitary group U .
- (d) The infinite unitary group is $U = \text{colim}_n U_n$. This is the union of all of the unitary groups U_n , including $U_n \hookrightarrow U_{n+1}$ by adding a row and a column to an $n \times n$ matrix and putting a 1 in the $(n+1, n+1)$ entry and zeros elsewhere.
- (e) This infinite unitary group is, surprisingly, an infinite loop space. This is a theorem known as *Bott Periodicity*: $U \simeq \Omega^2 U$. The spaces in between are $\Omega U \simeq \mathbb{Z} \times BU$. So we have a sequence of spaces

$$\mathbb{Z} \times BU, U, \mathbb{Z} \times BU, U, \dots$$

REFERENCES

- [Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.
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- [May99] J. P. May. *A concise course in algebraic topology*. Chicago, IL: University of Chicago Press, 1999.
- [Zak24] Inna Zakharevich. K-theory and Characteristic Classes: A homotopical perspective. <https://pi.math.cornell.edu/~zakh/book.pdf>, 2024.