

Reading: [Mal23, Sections 2.1, 2.2, and 2.3].

- (1) Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots$ be a sequence of spectra. Show that the homotopy colimit [Mal23, Definition 2.3.17] of this sequence commutes with stable homotopy groups, in the sense that

$$\pi_k(\operatorname{hocolim}_n X_n) = \operatorname{colim}_n \pi_k(X_n).$$

- (2) Eilenberg-MacLane spectra are characterized by their homotopy groups: if any other spectrum X satisfies $\pi_i X = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

- (a) Prove that $H(\mathbb{Z}/p)$ is stably equivalent to the homotopy cofiber of the map obtained by applying the functor H to $p: \mathbb{Z} \rightarrow \mathbb{Z}$.
- (b) The *rationalization* $S_{\mathbb{Q}}$ of the sphere spectrum S is the homotopy colimit of the diagram

$$S \xrightarrow{1} S \xrightarrow{2} S \xrightarrow{3} S \xrightarrow{4} S \xrightarrow{5} S \rightarrow \cdots$$

Prove that $S_{\mathbb{Q}}$ is stably equivalent to $H\mathbb{Q}$.

- (3) Let $\mathcal{Top}_*$ be the category of well-based spaces*. Let $\operatorname{ev}_0: \mathcal{Sp} \rightarrow \mathcal{Top}_*$ be the functor that evaluates a spectrum at its zeroth space: $\operatorname{ev}_0 X = X_0$.

- (a) Prove that Σ^∞ is left adjoint to ev_0 .
- (b) For any spectrum X , let $\Omega^\infty X := \operatorname{ev}_0 RX$, where R is the fibrant replacement functor [Mal23, Proposition 2.2.9]. Use [Rie14, Exercise 2.2.15] to prove that there is an adjunction

$$\Sigma^\infty: \operatorname{ho}(\mathcal{Top}_*) \rightleftarrows \operatorname{ho}(\mathcal{Sp}): \Omega^\infty.$$

- (4) Show that a fiber sequence of spectra $X \rightarrow Y \rightarrow Z$ yields a long exact sequence in homotopy groups.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.

*Recall that a pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. This guarantees that the reduced suspension of X is equivalent to the homotopy pushout of $* \leftarrow X \rightarrow *$.