

Reading: [Mal23, Sections 2.1, 2.2, and 2.3].

- (1) Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \dots$ be a sequence of spectra. Show that the homotopy colimit [Mal23, Definition 2.3.17] of this sequence commutes with stable homotopy groups, in the sense that

$$\pi_k(\operatorname{hocolim}_n X_n) = \operatorname{colim}_n \pi_k(X_n).$$

SOLUTION: We'll start with the right hand side and produce a chain of isomorphisms leading to the left hand side. Let $X_{n,m}$ be the m -th space in the spectrum X_n .

$$\begin{aligned} \operatorname{colim}_n \pi_k(X_n) &= \operatorname{colim}_n \operatorname{colim}_m \pi_{k+m}(X_{n,m}) && \text{definition of } \pi_k \text{ of a spectrum} \\ &\cong \operatorname{colim}_m \operatorname{colim}_n \pi_{k+m}(X_{n,m}) && \text{colimits commute with colimits} \\ &\cong \operatorname{colim}_m \pi_{k+m}(\operatorname{hocolim}_n X_{n,m}) && \text{the same fact for spaces (PSet 3, Q4)} \\ &\cong \pi_k(\operatorname{hocolim}_n X_n) && \text{definition of } \pi_k(\operatorname{hocolim}_n X_n) \end{aligned}$$

- (2) Eilenberg-MacLane spectra are characterized by their homotopy groups: if any other spectrum X satisfies $\pi_i X = 0$ for $i \neq 0$, then $X \simeq H(\pi_0 X)$.

- (a) Prove that $H(\mathbb{Z}/p)$ is stably equivalent to the homotopy cofiber of the map obtained by applying the functor H to $p: \mathbb{Z} \rightarrow \mathbb{Z}$.

SOLUTION: There is a homotopy cofiber sequence in spectra:

$$H\mathbb{Z} \xrightarrow{Hp} H\mathbb{Z} \rightarrow \operatorname{cof}(Hp)$$

We want to show that the cofiber $Q := \operatorname{cof}(Hp)$ has a single nonzero homotopy group concentrated in degree zero, and that this homotopy group is $\mathbb{Z}/p$. To do so, we break up the homotopy cofiber sequence of spectra into homotopy cofiber sequences of pointed spaces:

$$K(\mathbb{Z}, n) \xrightarrow{p} K(\mathbb{Z}, n) \rightarrow Q_n.$$

Ideally, we could compute the homotopy groups of Q_n from a long exact sequence in homotopy, but this is a cofiber sequence and not a fiber sequence. The idea now is to apply Blakers–Massey to get a fiber sequence to which we can apply the LES in homotopy.

Rewrite this cofiber sequence as a homotopy pushout of pointed spaces

$$\begin{array}{ccc} K(\mathbb{Z}, n) & \xrightarrow{p} & K(\mathbb{Z}, n) \\ \downarrow & & \downarrow \\ * & \longrightarrow & Q_n \end{array}$$

Then we can take the homotopy pullback R of $* \rightarrow Q_n \leftarrow K(\mathbb{Z}, n)$ and apply Blakers–Massey: the map $p: K(\mathbb{Z}, n) \rightarrow K(\mathbb{Z}, n)$ is $(n-1)$ -connected, and the map $K(\mathbb{Z}, n) \rightarrow *$ is n -connected, so the canonical map $K(\mathbb{Z}, n) \rightarrow R$ is $(n-1) + n - 1 = (2n-2)$ -connected.

$$\begin{array}{ccccc} K(\mathbb{Z}, n) & & \xrightarrow{p} & & K(\mathbb{Z}, n) \\ & \searrow & & \nearrow & \\ & & R & & \\ & \nearrow & \lrcorner_h & \searrow & \\ * & & & & Q_n \end{array}$$

Then we have a homotopy fiber sequence

$$R \rightarrow K(\mathbb{Z}, n) \rightarrow Q_n,$$

and $\pi_k(R) \cong \pi_k(K(\mathbb{Z}, n))$ for $0 \leq k < 2n - 2$. Thus, we learn that

$$\pi_k(Q_n) = \begin{cases} \mathbb{Z}/p & (k = n) \\ 0 & (0 \leq k < n) \\ 0 & (n < k < 2n - 2) \\ ? & (k \geq 2n - 2) \end{cases}$$

We can use this to compute the spectrum homotopy of $Q = \text{cof}(H\mathbb{Z} \xrightarrow{Hp} H\mathbb{Z})$

$$\pi_k Q = \text{colim}_n \pi_{k+n} Q_n$$

When $k = 0$, we learn that $\pi_k Q = \mathbb{Z}/p$, and when $k \neq 0$, we learn that $\pi_k Q = 0$. Therefore, Q has a single nonzero homotopy group concentrated in degree zero, so $Q \simeq H(\mathbb{Z}/p)$.

(b) The *rationalization* S_Q of the sphere spectrum S is the homotopy colimit of the diagram

$$S \xrightarrow{1} S \xrightarrow{2} S \xrightarrow{3} S \xrightarrow{4} S \xrightarrow{5} S \rightarrow \dots$$

Prove that S_Q is stably equivalent to HQ .

SOLUTION: The rationalization of S is the spectrum

$$S_Q = \text{hocolim}(S \xrightarrow{1} S \xrightarrow{2} S \xrightarrow{3} S \xrightarrow{4} \dots).$$

Recall that stable homotopy groups commute with homotopy colimits (turning the homotopy colimits to colimits in $\mathcal{A}b$). So,

$$\pi_n(S_Q) = \text{colim}(\pi_n S \xrightarrow{1} \pi_n S \xrightarrow{2} \pi_n S \xrightarrow{3} \dots)$$

This is isomorphic to $\pi_n S \otimes Q$. Recall from the last problem set that $\pi_n S$ is a finite abelian group for $n > 0$ and $\pi_n S = \mathbb{Z}$ for $n = 0$. Therefore, $\pi_n(S_Q) = \pi_n S \otimes Q = 0$ for $n > 0$ and $\pi_0(S_Q) = \mathbb{Z} \otimes Q = Q$. So this has one nonzero homotopy group, so $S_Q \simeq HQ$.

See also [Mal23, Chapter 1, Exercise 39] and [Mal23, Example 2.5.30].

(3) Let $\mathcal{Top}_*$ be the category of well-based spaces*. Let $\text{ev}_0 : \mathcal{Sp} \rightarrow \mathcal{Top}_*$ be the functor that evaluates a spectrum at its zeroth space: $\text{ev}_0 X = X_0$.

(a) Prove that Σ^∞ is left adjoint to ev_0 .

SOLUTION: We must show there is a natural isomorphism

$$\text{Hom}_{\mathcal{Sp}}(\Sigma^\infty X, Y) \cong \text{Hom}_{\mathcal{Top}_*}(X, Y_0)$$

Recall that a map of spectra $\Sigma^\infty \rightarrow Y$ is a sequence of pointed maps $f_n : \Sigma^n X \rightarrow Y_n$ such that the following squares commute for all n :

$$\begin{array}{ccc} \Sigma(\Sigma^n X) & \xrightarrow{\Sigma f_n} & \Sigma Y_n \\ \downarrow = & & \downarrow \\ \Sigma^{n+1} X & \xrightarrow{f_{n+1}} & Y_{n+1} \end{array}$$

*Recall that a pointed space X is *well-based* if the inclusion of the basepoint is a cofibration. This guarantees that the reduced suspension of X is equivalent to the homotopy pushout of $* \leftarrow X \rightarrow *$.

This is naturally determined by the map $f_0 : X \rightarrow Y_0$. Denote the composite structure maps of Y by $y_n : \Sigma^n Y_0 \rightarrow Y_n$. Note that the diagram above forces $f_1 = y_1 \circ \Sigma f_0$. Suppose for induction that $f_n = y_n \circ \Sigma f_0$, then the above square commuting forces $f_{n+1} = y_{n+1} \circ \Sigma f_0$. So the map $\Sigma^\infty X \rightarrow Y$ uniquely determines the map $f_0 : X \rightarrow Y_0$ by the structure maps of Y . Going the other way, any map $g : X \rightarrow Y_0$ can naturally ascend to a map of spectra $\Sigma^\infty X \rightarrow Y$ by setting $g_0 = g$ and $g_n = y_n \circ \Sigma g_0$. These constructions are clearly natural in X and Y , so this completes the proof of the adjunction.

- (b) For any spectrum X , let $\Omega^\infty X := \text{ev}_0 R X$, where R is the fibrant replacement functor [Mal23, Proposition 2.2.9]. Use [Rie14, Exercise 2.2.15] to prove that there is an adjunction

$$\Sigma^\infty : \text{ho}(\text{Top}_*) \rightleftarrows \text{ho}(\text{Sp}) : \Omega^\infty.$$

SOLUTION: First, observe that Σ^∞ is homotopical. For any space X , the stable homotopy groups of $\Sigma^\infty X$ are the homotopy groups of X , and if $X \xrightarrow{f} Y$ is a weak equivalence of spaces, the maps induced by $\Sigma^\infty f$ between the stable homotopy groups are exactly the isomorphisms induced by f between the homotopy groups of X and Y . Hence, Σ^∞ is a stable equivalence.

Note that the above is true so long as our spaces are well-based, so that $S^1 \wedge X$ agrees with the homotopy pushout of $* \leftarrow X \rightarrow *$. If the spaces are not well based, then these need not agree. See this mathOverflow answer: [?].

A derived functor for ev_0 can be computed by pre-composing with a right deformation of Sp . In particular, the functor R in the exercise is such a deformation. ev_0 is homotopical on the full subcategory of Sp generated by the image of R : if $f : X \rightarrow Y$ is a stable equivalence between two Ω -spectra, then $f_0 : X_0 \rightarrow Y_0$ is a weak equivalence because $\pi_m \cong \pi_m(X_0)$, $\pi_m(Y) \cong \pi_m(Y_0)$ and since $\{f_m\}_{m \in \mathbb{N}}$ commute with the bonding maps, the isomorphisms $\pi_m(X) \rightarrow \pi_m(Y)$ induced by f is exactly the map induced by f_0 from $\pi_m X_0$ to $\pi_m(Y_0)$.

So $\Omega^\infty = \text{ev}_0 \circ R$ is a right derived functor for ev_0 . By proposition 2.2.13 in Riehl, a derived functor computed thanks to a deformation is an absolute Kan extension. Therefore, by [Rie14, Exercise 2.2.15], Σ^∞ and Ω^∞ are an adjoint pair.

- (4) Show that a fiber sequence of spectra $X \rightarrow Y \rightarrow Z$ yields a long exact sequence in homotopy groups.

SOLUTION: This exercise asks you to fill out the details in [Mal23, Lemma 2.4.9]. A sketch of the solution can be found in the proof of that lemma.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Rie14] Emily Riehl. *Categorical homotopy theory*, volume 24 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2014.