

# STABILITY TALK OUTLINE

Read [Mal23, Sections 2.4 and 2.5]. The main point of this talk is to discuss the ways in which  $\mathcal{S}p$  is a nicer category than the category of spaces – namely,  $\mathrm{ho}(\mathcal{S}p)$  has the property of *stability* but  $\mathrm{ho}(\mathcal{T}op_*)$  does not.

## OUTLINE

### (1) Overview and Recollections

- (a) Recall the definition of spectra and maps of spectra [Mal23, Definition 2.1.1]. The category of spectra and the maps described above is denoted  $\mathcal{S}p$ .
- (b) Spectra are better than spaces insofar as  $\mathrm{ho}(\mathcal{S}p)$  is a stable category.
- (c) There are three different ways to formulate stability:
  - $\Sigma$  and  $\Omega$  are inverses up to stable equivalence.
  - Cofiber sequences and fiber sequences are the same, up to stable equivalence.
  - Homotopy pushout squares are homotopy pullback squares, up to stable equivalence.
- (d) We will recall definitions of cofiber sequences, fiber sequences, homotopy pushouts, and homotopy pullbacks as we need them.
- (e) Give the warning that although homotopy pushouts are homotopy pullbacks, it is not true that the homotopy colimit of a diagram is the same as the homotopy limit of the same diagram! Finding a counterexample is a homework exercise.

### (2) Stability as an equivalence $\Sigma^{-1} \simeq \Omega$ .

- (a) Recall the suspension  $\Sigma X$  and loops  $\Omega X$  of a spectrum  $X$  [Mal23, Definitions 2.3.6 and 2.3.8]. Do so without reference to the smash  $K \wedge X$  or the function spectrum  $F(K, X)$ , as much as is possible.
- (b) Give a quick sketch argument as to why  $\Sigma$  is left adjoint to  $\Omega$  as functors  $\mathcal{S}p \rightarrow \mathcal{S}p$ : these functors are adjoints on  $\mathcal{T}op_*$ , and we're just doing those operations levelwise on spectra.
- (c) State [Mal23, Proposition 2.4.1] and prove it.
- (d) State [Mal23, Corollary 2.4.5] and prove it. Note that this doesn't reference or require [Mal23, Corollary 2.4.3] – you can skip that one.
- (e) The consequence is that  $\Sigma$  and  $\Omega$  are inverse functors up to stable equivalence.

### (3) Stability in terms of fiber and cofiber sequences

- (a) Define fiber and cofiber sequences of spectra following [Mal23, Definition 2.4.6]. You will need to define a homotopy of maps of spectra too [Mal23, Definition 2.3.11].
- (b) Prove that any fiber sequence of spectra yields a LES in homotopy [Mal23, Lemma 2.4.9].
- (c) Prove that any cofiber sequence of spectra yields a LES in homotopy [Mal23, Lemma 2.4.10]. Make a note that this is not true for spaces!
- (d) Sketch the proof that  $\Sigma \mathrm{hofib}(f) \simeq \mathrm{hocof}(f)$ , where  $\mathrm{hofib}(f)$  and  $\mathrm{hocof}(f)$  are homotopy fiber and cofiber of  $f$  [Mal23, 2.4.12]. It's probably best if you don't go too much into the details, but just indicate the construction of the map and draw the diagram and sketch the proof from there.
- (e) Prove that a sequence is a fiber sequence of spectra if and only if it is a cofiber sequence of spectra [Mal23, Proposition 2.4.13]. This is the big theorem here!

(4) Stability in terms of homotopy pushouts and pullbacks

- (a) Define homotopy pushout square and homotopy pullback square following [Mal23, Definition 2.4.14].
- (b) State but do not prove [Mal23, Lemma 2.4.15]. It will be an exercise on the homework.
- (c) Use the lemma to prove [Mal23, Corollary 2.4.16]. This is perhaps the strongest of the versions of stability introduced at the beginning of the talk.
- (d) Explore some consequences of this last theorem: show that [Mal23, Corollary 2.4.16] implies both [Mal23, Proposition 2.4.13] and [Mal23, Corollary 2.4.5]. To see the latter, note that the homotopy pushout of  $* \leftarrow X \rightarrow *$  is  $\Sigma X$  and the homotopy pullback of  $* \leftarrow X \rightarrow *$  is  $\Omega X$ , and consider the homotopy pushout square below (and its dual).

$$\begin{array}{ccc} X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

- (e) Show that coproducts and products of spectra are weakly equivalent, following [Mal23, Proposition 2.4.19].
- (5) Extra stuff to think about, if you have time or energy left after the above
- (a) Now that you know how to construct cofibers, you can define the spectrum  $S/p$  [Mal23, Example 2.5.29]. This is the cofiber of the map  $\Sigma^{-1}\Sigma^\infty p: S \rightarrow S$ , where  $p: S^1 \rightarrow S^1$  is a degree  $p$  map. Explain how this is related to the universal coefficient sequence.
  - (b) Define the rationalization of the sphere spectrum  $S_{\mathbb{Q}}$  [Mal23, Example 2.5.30]. By a homework problem, we know that  $S_{\mathbb{Q}}$  is equivalent to  $H\mathbb{Q}$ . Prove this! (it should be short)
  - (c) Define  $p$ -localization of a spectrum, and  $p$ -completion [Mal23, Examples 2.5.31, 2.5.32]. This can be a quick aside after rationalization.
  - (d) Explain how to rationalize or take any spectrum mod- $p$  [Mal23, Example 2.5.33].
  - (e) State the theorem that the rationalization of any spectrum is just a sum of shifted copies of  $H\mathbb{Q}$ , cf. [Mal23, Example 2.5.34]. Give the example for rationalized K-theory  $K\mathbb{U}_{\mathbb{Q}}$ .

## REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. [http://people.math.binghamton.edu/malkiewich/spectra\\_book\\_draft.pdf](http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf), October 2023.