

Reading: [Mal23, Sections 2.4 and 2.5].

- (1) Consider the commuting square of spectra

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ C & \xrightarrow{g} & D \end{array}$$

- (a) Prove that this square is a homotopy pushout if and only if the induced map of homotopy cofibers $\mathrm{cof}(f) \rightarrow \mathrm{cof}(g)$ is a stable equivalence. Dually, this square is a pullback if and only if the induced map $\mathrm{fib}(f) \rightarrow \mathrm{fib}(g)$ is an isomorphism.
- (b) Use this fact to prove that a commuting square of spectra is a homotopy pullback if and only if it is a homotopy pushout. You may use the fact that the suspension and loops functors are homotopy inverses: $\Sigma^{-1} \simeq \Omega$.
- (2) A *semiadditive category* is a category $\mathcal{A}$ with a zero object 0 that admits all finite products and coproducts, such that the canonical morphism $X \amalg Y \rightarrow X \times Y$ is an isomorphism.

- (a) Show that $\mathrm{ho}(\mathcal{S}p)$ is a semiadditive category.

An *additive functor* between semiadditive categories is a functor that preserves the zero object and preserves finite products/coproducts.

- (b) Is the Eilenberg–MacLane spectrum functor $H: \mathcal{A}b \rightarrow \mathrm{ho}(\mathcal{S}p)$ an additive functor?
- (3) Let $k \geq 0$. Define the *shift functor* $\mathrm{sh}_k: \mathcal{S}p \rightarrow \mathcal{S}p$ by $\mathrm{sh}_k(X)_n = X_{k+n}$.

- (a) Prove that there is a natural stable equivalence $\Sigma \simeq \mathrm{sh}_1$.
- (b) Define functors sh_k for $k < 0$. Prove that $\mathrm{sh}_{-1} \simeq \Omega$.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [nLa] nLab. Pasting law. <https://ncatlab.org/nlab/show/pasting+law+for+pullbacks>.