

Reading: [Mal23, Section 2.6].

- (1) Recall that the *free spectrum functor* $F_n: \mathcal{Top}_* \rightarrow \mathcal{Sp}$ can be described by $F_n K \simeq \Sigma^{-n} \Sigma^\infty K$ for any $n \in \mathbb{Z}$.
 - (a) When $n \geq 0$, prove that F_n is left adjoint to evaluation $ev_n: \mathcal{Sp} \rightarrow \mathcal{Top}_*$, where ev_n is the functor that takes the n -th space of a spectrum: $ev_n X = X_n$.
 - (b) Does F_n have a right adjoint when $n < 0$?
- (2) Consider the homotopy pushout/pullback square of spectra:

$$\begin{array}{ccc} X & \xrightarrow{f} & B \\ \downarrow f' & & \downarrow g' \\ A & \xrightarrow{g} & Y. \end{array}$$

Prove that there is a Mayer–Vietoris-type long exact sequence of spectra:

$$\cdots \rightarrow \pi_{n+1} Y \rightarrow \pi_n X \rightarrow \pi_n A \oplus \pi_n B \rightarrow \pi_n Y \rightarrow \pi_{n-1} X \rightarrow \cdots$$

- (3) A spectrum X is called **n -connected** if $\pi_i X = 0$ for $i \leq n$, or **n -connective** if $\pi_i X = 0$ for $i < n$. Let $X \rightarrow Y \rightarrow Z$ be a cofiber/fiber sequence of spectra.
 - (a) Prove that if X and Z are n -connected, then so is Y .
 - (b) What can you say about connectivity of Z if X and Y are n -connected? What can you say about connectivity of X if Y and Z are n -connected?
- (4) Prove that the following three conditions are equivalent:
 - (a) X is a *finite spectrum*, i.e. X is stably equivalent to a cellular spectrum with finitely many stable cells.
 - (b) X is stably equivalent to a free spectrum $F_k K \simeq \Sigma^{-k} \Sigma^\infty K$ for a finite cell complex K .
 - (c) X is bounded below and the direct sum of the homology groups $\bigoplus_k H_k(X; \mathbb{Z})$ is finitely generated as an abelian group.

REFERENCES

[Hat02] Allen Hatcher. *Algebraic topology*. Cambridge: Cambridge University Press, 2002.

[Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.