

Reading: [Mal23, Section 3.2] and [Wei94, Section 10.2]

- (1) A spectrum X is *rational* if each of its homotopy groups is a $\mathbb{Q}$ -vector space. Let $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ be the full subcategory of $\mathrm{ho}(\mathcal{S}p)$ whose objects are rational spectra. Prove that $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ is a triangulated subcategory of $\mathrm{ho}(\mathcal{S}p)$.

SOLUTION: Since the homotopy groups of a finite product/wedge of spectra are just the products of the homotopy groups of its summands, we know that $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ is additive. Moreover, since Σ only affects the homotopy groups of a spectrum by shifting their degrees, we know that Σ restricts to an equivalence $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}} \rightarrow \mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$.

We are left with checking that the cofiber of a map of rational spectra is rational. Let $f : X \rightarrow Y$ be a map of rational spectra, and let $C = \mathrm{cof}(f)$. We have a long exact sequence in homotopy, from which we can extract a bunch of short exact sequences of the form:

$$0 \rightarrow \ker \delta_n \rightarrow \pi_n C \rightarrow \mathrm{im} \delta_n \rightarrow 0$$

where $\delta_n : \pi_n C \rightarrow \pi_{n-1} X$ is the connecting homomorphism. By exactness, we have

$$\ker \delta_n \cong \pi_n Y / \mathrm{im}(\pi_n X \rightarrow \pi_n Y),$$

which is a quotient of a $\mathbb{Q}$ -vector space by a $\mathbb{Q}$ -vector subspace, and thus rational. Similarly $\mathrm{im} \delta_n \subseteq \pi_{n-1} X$ and is thus a rational vector space. By choosing bases for these, we obtain isomorphisms

$$\pi_n C \cong \ker \delta_n \oplus \mathrm{im} \delta_n$$

for all n , showing that $\pi_n C$ is rational for all n and thus $C = \mathrm{cof}(f)$ is rational. Therefore $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ is a triangulated subcategory of $\mathrm{ho}(\mathcal{S}p)$.

- (2) Define a triangulated functor $H : \mathcal{D}(\mathbb{Q}) \rightarrow \mathrm{ho}(\mathcal{S}p)$ such that $\pi_n H(V_{\bullet}) = H_n(V_{\bullet})$ for all $n \in \mathbb{Z}$. *Hint: any chain complex of $\mathbb{Q}$ -vector spaces is quasi-isomorphic to its homology.*

SOLUTION: Given a chain complex $V_{\bullet}$, define $H(V_{\bullet}) = \bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(V_{\bullet}))$. That is, H takes a chain complex $V_{\bullet}$ to the wedge sum of shifts of the Eilenberg–MacLane spectra of its homology groups. Since quasi-isomorphisms induce isomorphisms on homology, this functor sends quasi-isos to weak equivalences. Therefore, it is a homotopical functor and descends to a functor $H : \mathcal{D}(\mathbb{Q}) \rightarrow \mathrm{ho}(\mathcal{S}p)$. Note that the image of this functor is contained in the full subcategory $\mathrm{ho}(\mathcal{S}p)_{\mathbb{Q}}$ spanned by the rational spectra.

To show that this functor is triangulated, we must show that it is additive, that it commutes with translation, and it preserves triangles.

To see that this functor is additive, first note that H applied to the zero chain complex is just a point.

Then:

$$\begin{aligned}
H(V_\bullet \oplus W_\bullet) &= \bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(V_\bullet \oplus W_\bullet)) \\
&\cong \bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(V_\bullet) \oplus H_n(W_\bullet)) && \text{homology distributes over products} \\
&\cong \bigvee_{n \in \mathbb{Z}} \Sigma^n (H(H_n(V_\bullet)) \vee H(H_n(W_\bullet))) && \text{Eilenberg-MacLane functor is additive (HW 5)} \\
&\cong \bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(V_\bullet)) \vee \Sigma^n H(H_n(W_\bullet)) && \text{suspension commutes with coproducts} \\
&\cong \left(\bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(V_\bullet)) \right) \vee \left(\bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(W_\bullet)) \right) \\
&= H(V_\bullet) \vee H(W_\bullet)
\end{aligned}$$

So H is an additive functor.

To see that H commutes with translation, recall that translation in the derived category is shifting the chain complex by 1: $V_\bullet \mapsto V_{\bullet+1}$. Then

$$\begin{aligned}
H(V_{\bullet+1}) &= \bigvee_{k \in \mathbb{Z}} \Sigma^k H(H_k(V_{\bullet+1})) \\
&= \bigvee_{k \in \mathbb{Z}} \Sigma^k H(H_{k-1}(V_\bullet)) \\
&= \bigvee_{n \in \mathbb{Z}} \Sigma^{n+1} H(H_n(V_\bullet)) && \text{reindexing with } n = k - 1 \\
&= \Sigma \left(\bigvee_{n \in \mathbb{Z}} \Sigma^n H(H_n(V_\bullet)) \right) \\
&= \Sigma H(V_\bullet)
\end{aligned}$$

So H commutes with translation.

Finally, we must check that H preserves distinguished triangles. Consider a distinguished triangle in $\mathcal{D}(Q)$, which is quasi-isomorphic to one of the form

$$V_\bullet \xrightarrow{f_\bullet} W_\bullet \rightarrow \text{cone}(f_\bullet) \rightarrow V_{\bullet+1}$$

We want to show that

$$H(V_\bullet) \xrightarrow{H(f_\bullet)} H(W_\bullet) \rightarrow H(\text{cone}(f_\bullet))$$

is a cofiber sequence in $\text{ho}(\mathcal{S}p)$. Consider the morphism g from the cofiber of $H(f_\bullet)$ to $H(\text{cone}(f_\bullet))$:

$$\begin{array}{ccccc}
H(V_\bullet) & \xrightarrow{H(f_\bullet)} & H(W_\bullet) & \longrightarrow & \text{cof}(H(f_\bullet)) \\
\parallel & & \parallel & & \downarrow g \\
H(V_\bullet) & \xrightarrow{H(f_\bullet)} & H(W_\bullet) & \longrightarrow & H(\text{cone}(f_\bullet))
\end{array}$$

Taking homotopy, we get a diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \pi_{n+1} \text{cof}(H(f_\bullet)) & \longrightarrow & \pi_n H(V_\bullet) & \longrightarrow & \pi_n H(W_\bullet) & \longrightarrow & \pi_n \text{cof}(H(f_\bullet)) & \longrightarrow & \pi_{n-1} H(V_\bullet) & \longrightarrow & \cdots \\
& & \downarrow g_* & & \parallel & & \parallel & & \downarrow g_* & & \parallel & & \\
\cdots & \longrightarrow & \pi_{n+1} H(\text{cone}(f_\bullet)) & \longrightarrow & \pi_n H(V_\bullet) & \longrightarrow & \pi_n H(W_\bullet) & \longrightarrow & \pi_n H(\text{cone}(f_\bullet)) & \longrightarrow & \pi_{n-1} H(V_\bullet) & \longrightarrow & \cdots
\end{array}$$

The top row is exact because it's the LES of a cofiber sequence. The bottom row is exact because it's isomorphic to the LES in homology for the distinguished triangle $V_\bullet \xrightarrow{f_\bullet} W_\bullet \rightarrow \text{cone}(f_\bullet)$ in $\mathcal{D}(\mathcal{Q})$. Then the five lemma shows that $\text{cof}(H(f_\bullet))$ is stably equivalent to $H(\text{cone}(f_\bullet))$. Therefore, H sends triangles to triangles.

We have shown that H is additive, commutes with shift functors, and sends triangles to triangles. So H is a triangulated functor.

- (3) Show that the class of stable equivalences is saturated in $\mathcal{S}p$.

SOLUTION: Recall that a homotopical category $\mathcal{C}$ is saturated if it has the following property: if $f \in \mathcal{C}$ becomes an isomorphism in $\text{ho}(\mathcal{C})$, then f is a weak equivalence.

So the question is asking us to show that if $f: X \rightarrow Y$ becomes an isomorphism in $\text{ho}(\mathcal{S}p)$, then f is a stable equivalence.

Let grAb denote the category of graded abelian groups, and consider it as a homotopical category equipped with the minimal homotopical structure (ie only isomorphisms are weak equivalences). Then $\pi_*: \mathcal{S}p \rightarrow \text{grAb}$ is a homotopical functor, by definition. Hence, it descends to a functor $\text{ho}(\mathcal{S}p) \rightarrow \text{grAb}$, and the triangle below commutes:

$$\begin{array}{ccc} \mathcal{S}p & \xrightarrow{\pi_*} & \text{grAb} \\ \downarrow & \nearrow \text{ho}(\pi_*) & \\ \text{ho}(\mathcal{S}p) & & \end{array}$$

Suppose that f is an isomorphism in $\text{ho}(\mathcal{S}p)$. Then $\text{ho}(\pi_*)(f)$ is an isomorphism in grAb , so f must have always been a stable equivalence.

- (4) (a) Show that in a triangulated category $\mathcal{C}$, all monomorphisms are split monomorphisms.

SOLUTION: Given a monic $f: A \rightarrow B$, we may extend to an exact triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$$

and by rotating the triangle we get another

$$B \xrightarrow{g} C \xrightarrow{h} \Sigma A \xrightarrow{-\Sigma f} \Sigma B.$$

It follows that $-\Sigma f \circ h = 0$ and thus that $f \circ \Sigma^{-1}h = 0$ and thus that $\Sigma^{-1}h = 0$ and $h = 0$ since f is monic. It follows that $B \cong A \oplus C$ so that f is split.

- (b) Show that the triangulated category $\mathcal{D}(\mathbb{Z})$ is not an abelian category.

SOLUTION: Assume for sake of contradiction that $\mathcal{D}(\mathbb{Z})$ is abelian. Consider the quotient homomorphism $\mathbb{Z}/4 \rightarrow \mathbb{Z}/2$, considered as chain complexes concentrated in degree zero inside $\mathcal{D}(\mathbb{Z})$. Since we are assuming $\mathcal{D}(\mathbb{Z})$ is abelian, this has a kernel, which is a monomorphism $f: K \hookrightarrow \mathbb{Z}/4$. In the triangulated category $\mathcal{D}(\mathbb{Z})$, all monomorphisms are split, so f has a left inverse $g: \mathbb{Z}/4 \rightarrow K$ such that $gf = \text{id}_K$.

The exact triangle $K \rightarrow \mathbb{Z}/4 \rightarrow \mathbb{Z}/2$ yields a long exact sequence

$$\cdots \rightarrow H_1(\mathbb{Z}/2) \rightarrow H_0(K) \xrightarrow{H_0(f)} H_0(\mathbb{Z}/4) \rightarrow H_0(\mathbb{Z}/2) \rightarrow H_{-1}(K) \rightarrow \cdots$$

This simplifies to the short exact sequence

$$0 \rightarrow H_0(K) \xrightarrow{H_0(f)} \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0.$$

Hence $H_0(K) \cong \mathbb{Z}/2$. Since f has a left inverse g , then $H_0(f)$ has a left inverse $H_0(g)$, which implies that the SES above splits. This is a contradiction.

(c) Show that $\mathrm{ho}(\mathcal{S}p)$ is not an abelian category.

SOLUTION: The strategy here is to apply the Eilenberg–MacLane spectrum functor H to the previous example.

Assume for sake of contradiction that $\mathrm{ho}(\mathcal{S}p)$ is abelian. Consider the map $H(\mathbb{Z}/4) \rightarrow H(\mathbb{Z}/2)$ induced by the quotient homomorphism $\mathbb{Z}/4 \rightarrow \mathbb{Z}/2$. The kernel of this in $\mathrm{ho}(\mathcal{S}p)$ is a spectrum K with a monomorphism $f: K \hookrightarrow H(\mathbb{Z}/4)$. Since $\mathrm{ho}(\mathcal{S}p)$ is triangulated, this monomorphism is split: there is a left inverse $g: H(\mathbb{Z}/4) \rightarrow K$ such that $gf = \mathrm{id}_K$.

The exact triangle $K \rightarrow H(\mathbb{Z}/4) \rightarrow H(\mathbb{Z}/2)$ is a fiber sequence in spectra, and yields a long exact sequence

$$\cdots \rightarrow \pi_1 H(\mathbb{Z}/2) \rightarrow \pi_0 K \xrightarrow{\pi_0 f} \pi_0 H(\mathbb{Z}/4) \rightarrow \pi_0 H(\mathbb{Z}/2) \rightarrow \pi_{-1} K \rightarrow \cdots$$

which simplifies to the short exact sequence

$$0 \rightarrow \pi_0 K \xrightarrow{\pi_0 f} \mathbb{Z}/4 \rightarrow \mathbb{Z}/2 \rightarrow 0$$

Hence, $\pi_0 K \cong \mathbb{Z}/2$. Since f has a left inverse g , $\pi_0 f$ has a left inverse $\pi_0 g$, which implies that the SES above splits. This is a contradiction, so $\mathrm{ho}(\mathcal{S}p)$ cannot be abelian.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Wei94] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.