

SYMMETRIC/ORTHOGONAL SPECTRA TALK OUTLINE

Read [Mal23, Sections 6.1 and 6.2]. For more on symmetric spectra, see [Sch07]. For more on orthogonal spectra, see [Sch23] (take $G = e$). [Dug22, Page 4] has a good table comparing the advantages and disadvantages of the various categories of spectra.

OUTLINE

(1) Symmetric and Orthogonal Spectra

- (a) Explain that the category of spectra $\mathcal{S}p$ that we have been working with so far is not exactly the right category of spectra. It doesn't have a smash product. We refer to $\mathcal{S}p$ as the category of *sequential spectra*.
- (b) Define symmetric spectra and the category $\mathcal{S}p^{\Sigma}$ [Mal23, Definition 6.1.2]
- (c) Define orthogonal spectra and the category $\mathcal{S}p^O$ [Mal23, Definition 6.1.3]
- (d) Explain that there are forgetful functors $\mathcal{S}p^O \rightarrow \mathcal{S}p^{\Sigma} \rightarrow \mathcal{S}p$.
- (e) Define the categories $\mathcal{S}$, $\mathcal{S}^{\Sigma}$, and $\mathcal{J} = \mathcal{S}^O$. Explain how we can view sequential spectra, symmetric spectra, and orthogonal spectra as functors out of these categories. See [Mal23, Definition 5.3.30, Lemma 5.3.32]. Introduce the term *diagram spectra* as in [Mal23, paragraph at the start of Section 6.1.4].
- (f) Define the free spectra $F_n^{\Sigma}(K) := K \wedge \mathcal{S}^{\Sigma}(n, -)$ and $F_n^O(K) := K \wedge \mathcal{S}^O(n, -)$, as in [Mal23, Definition 5.3.33].
- (g) Use these to describe the right adjoints to the forgetful functors.
- (h) Describe how to find (homotopy) limits and (homotopy) colimits in the categories of symmetric and orthogonal spectra [Mal23, Remark 6.1.16].
- (i) State [Mal23, Theorem 6.1.26] and its corollary, and pick a few items to prove. Don't prove all of them. The point is that sequential and symmetric spectra work basically the same as our sequential spectra from before.

(2) Equivalence of $ho(\mathcal{S}p^{\Sigma})$ and $ho(\mathcal{S}p)$ and the difficulty with weak equivalences in symmetric spectra

- (a) Define homotopy groups of symmetric and orthogonal spectra and define a π_* -isomorphism [Mal23, Definition 6.1.17].
- (b) Explain how a π_* -isomorphism in symmetric spectra doesn't capture the right notion of stable equivalence.
- (c) Define a stable equivalence of diagram spectra [Mal23, Definition 6.1.18].
- (d) Explain that stable equivalences and π_* -isomorphisms are equivalent in sequential spectra.
- (e) Prove that a π_* -isomorphism is a stable equivalence for all three types of spectra [Mal23, Proposition 6.1.24].
- (f) Define the *true homotopy groups* of a symmetric spectrum as the homotopy groups of $\pi_*(RX)$, where R is the fibrant replacement functor [Mal23, Definition 6.1.35]. Explain that a map of symmetric spectra is a stable equivalence if it induces an isomorphism on the true homotopy groups.
- (g) State the theorem that π_* -isomorphisms and stable equivalences are the same for orthogonal spectra [Mal23, Proposition 6.1.32]. You don't have to prove it.

- (h) State the theorem that with the stable equivalences, $\mathrm{ho}(\mathrm{Sp}) \simeq \mathrm{ho}(\mathrm{Sp}^{\Sigma}) \simeq \mathrm{ho}(\mathrm{Sp}^{\mathrm{O}})$ [Mal23, Proposition 6.1.34]. You don't have to prove it.
- (3) Constructing the smash product
- (a) Define the smash product of symmetric spectra [Mal23, Definition 6.2.1]. Indicate how it works for orthogonal spectra too.
 - (b) Give a few examples.
 - (c) Sketch the proof of [Mal23, Proposition 6.2.6], and state [Mal23, Theorem 6.2.8].
 - (d) State [Mal23, Theorem 6.2.22] and use it to prove [Mal23, Lemma 6.2.24]. Explain how to use this to prove that Sp^{Σ} and Sp^{O} are symmetric monoidal categories. Sketch the proof of Theorem 6.2.22 only if you think you'll have time.
- (4) Ring spectra
- (a) Explain how the smash product has a universal property similar to the universal property of the tensor product coming from bilinear homomorphisms. [Mal23, Definition 6.2.9].
 - (b) Define a ring spectrum [Mal23, Definition 6.2.26] and explain the remark [Mal23, Remark 6.2.27]: a ring spectrum structure is equivalent to a collection of maps $R_q \wedge R_p \rightarrow R_{p+q}$ that commute with bonding maps. See also [Sch07, Definition 1.3] for precisely which diagrams must commute.
- (5) Extra things that you can do if you have the interest or time
- (a) Describe the internal hom in Sp^{Σ} according to [Mal23, Section 6.2.2], and show that the function spectra $F(X, -)$ are right adjoint to the smash product $X \wedge -$.

REFERENCES

- [Dug22] Daniel Dugger. Stable categories and spectra via model categories. In *Stable categories and structured ring spectra*, pages 75–150. Cambridge: Cambridge University Press, 2022.
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