

SYMMETRIC & ORTHOGONAL SPECTRA NAME: _____

Reading: [Mal23, Sections 6.1 and 6.2] and [Sch23, Section 1].

- (1) Let X be a symmetric spectrum such that Σ_n acts trivially on X_n for all n .
 - (a) Prove that the orbit space $(S^n)_{\Sigma_n}$ is contractible for all $n \geq 2$, with Σ_n -action on S^n by permutation of coordinates, viewing S^n as the one-point compactification of $\mathbb{R}^n$.
 - (b) Show that the (naive) homotopy groups of X are trivial.

- (2) A *symmetric ring spectrum* is a symmetric spectrum R together with $\Sigma_n \times \Sigma_m$ -equivariant multiplication maps $\mu_{n,m}: R_n \wedge R_m \rightarrow R_{n+m}$ and unit maps $\iota_0: S^0 \rightarrow R_0$ and $\iota_1: S^1 \rightarrow R_1$ satisfying associativity, unit, multiplicativity, and centrality conditions (see [Sch07, Definition 1.3]).

Show that a symmetric ring spectrum in the sense above is a monoid in the monoidal category of symmetric spectra, with smash product as in [Mal23, Definition 6.2.1].

- (3) Cobordism of manifolds is captured by the spectrum MO . Read about this spectrum in [Mal23, Example 2.1.20] and [Sch07, Example 2.8].
 - (a) Prove that there is a pullback square of vector bundles

$$\begin{array}{ccc} \gamma_n \oplus \gamma_m & \longrightarrow & \gamma_{n+m} \\ \downarrow & & \downarrow \\ BO(n) \times BO(m) & \longrightarrow & BO(n+m), \end{array}$$

where $\gamma_k \rightarrow BO(k)$ is the tautological bundle.

- (b) Use the pullback square to produce multiplication maps $\mu_{n,m}: MO(n) \wedge MO(m) \rightarrow MO(n+m)$ for all $n, m \geq 0$.
- (c) Define unit maps $\iota_0: S^0 \rightarrow MO(0)$ and $\iota_1: S^1 \rightarrow MO(1)$.
- (d) Show that these maps make MO into a commutative ring orthogonal spectrum.

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.
- [Sch07] Stefan Schwede. An untitled book project about symmetric spectra. <http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf>, 2007.
- [Sch23] Stefan Schwede. Lectures on equivariant stable homotopy theory. <http://www.math.uni-bonn.de/people/schwede/equivariant.pdf>, 2023.