

Reading: [Mal23, Sections 6.1 and 6.2].

(1) Let $F \dashv G$ be an adjoint pair of functors between monoidal categories.

(a) Show that if F is strong symmetric monoidal, then G is lax symmetric monoidal.

SOLUTION: Let $F \dashv G$ be an adjunction between symmetric monoidal categories $\mathcal{C}$ and $\mathcal{D}$. We will only assume that F is oplax (we only ask for the “inverse” of the structure maps, we do not need the forward direction) i.e. that there are transformations

$$d: F(I) \rightarrow I \text{ and } n: F(A \otimes B) \rightarrow F(A) \otimes F(B)$$

which are compatible with the associativity, unit, and symmetry isomorphisms. We will show that G has an associated lax structure i.e. that there are transformations

$$e: I \rightarrow G(I) \text{ and } m: G(A) \otimes G(B) \rightarrow G(A \otimes B)$$

with similar compatibility.

The map e is the adjunct of d , meaning the image of d under the isomorphism $\mathcal{C}(F(I), I) \cong \mathcal{D}(I, G(I))$, but I find it hard to prove the axioms with this definition, and m is not just the adjunct of n . Let $\eta: \text{id} \Rightarrow GF$ and $\epsilon: FG \Rightarrow 1$ be the unit and counit of the adjunction. Then, e is the composite

$$e: I \xrightarrow{\eta} G(F(I)) \xrightarrow{G(d)} G(I).$$

Similarly, m is the composite

$$m: G(A) \otimes G(B) \xrightarrow{\eta} G(F(G(A) \otimes G(B))) \xrightarrow{G(n)} G(F(G(A)) \otimes F(G(B))) \xrightarrow{G(\epsilon \otimes \epsilon)} G(A \otimes B).$$

It remains to show that these new transformations have the right compatibility.

To show compatibility with the symmetry, we need to show that if

$$\begin{array}{ccc} F(A \otimes B) & \xrightarrow{n} & F(A) \otimes F(B) \\ \downarrow F(\sigma) & & \downarrow \sigma \\ F(B \otimes A) & \xrightarrow{n} & F(B) \otimes F(A) \end{array} \quad \text{commutes then} \quad \begin{array}{ccc} G(A) \otimes G(B) & \xrightarrow{m} & G(A \otimes B) \\ \downarrow \sigma & & \downarrow G(\sigma) \\ G(B) \otimes G(A) & \xrightarrow{m} & G(B \otimes A) \end{array}$$

commutes. Expanding the right diagram, we are tasked with showing that

$$\begin{array}{ccc} G(A) \otimes G(B) & \xrightarrow{\eta} G(F(G(A) \otimes G(B))) & \xrightarrow{G(n)} G(F(G(A)) \otimes F(G(B))) & \xrightarrow{G(\epsilon \otimes \epsilon)} G(A \otimes B) \\ \downarrow \sigma & & & \downarrow G(\sigma) \\ G(B) \otimes G(A) & \xrightarrow{\eta} G(F(G(B) \otimes G(A))) & \xrightarrow{G(n)} G(F(G(B)) \otimes F(G(A))) & \xrightarrow{G(\epsilon \otimes \epsilon)} G(B \otimes A) \end{array}$$

commutes which we show by filling in the squares.

$$\begin{array}{ccccccc} G(A) \otimes G(B) & \xrightarrow{\eta} & G(F(G(A) \otimes G(B))) & \xrightarrow{G(n)} & G(F(G(A)) \otimes F(G(B))) & \xrightarrow{G(\epsilon \otimes \epsilon)} & G(A \otimes B) \\ \downarrow \sigma & & \downarrow G(F(\sigma)) & & \downarrow G(\sigma) & & \downarrow G(\sigma) \\ G(B) \otimes G(A) & \xrightarrow{\eta} & G(F(G(B) \otimes G(A))) & \xrightarrow{G(n)} & G(F(G(B)) \otimes F(G(A))) & \xrightarrow{G(\epsilon \otimes \epsilon)} & G(B \otimes A) \end{array}$$

The outside squares commute by the naturality of η and σ , and the middle square is the compatibility square for F plugged into G .

To show compatibility with the (left) unit λ , we need to show that if

$$\begin{array}{ccc} F(I \otimes A) & \xrightarrow{\eta} & F(I) \otimes F(A) \\ \downarrow F(\lambda) & & \downarrow d \otimes 1 \\ F(A) & \xleftarrow{\lambda} & I \otimes F(A) \end{array} \quad \text{commutes then} \quad \begin{array}{ccc} I \otimes G(A) & \xrightarrow{e \otimes 1} & G(I) \otimes G(A) \\ \downarrow \lambda & & \downarrow m \\ G(A) & \xleftarrow{G(\lambda)} & G(I \otimes A) \end{array}$$

commutes. Expanding the right diagram, we are tasked to show that

$$\begin{array}{ccccc} I \otimes G(A) & \xrightarrow{\eta \otimes 1} & G(F(I)) \otimes G(A) & \xrightarrow{G(d) \otimes 1} & G(I) \otimes G(A) \\ \downarrow \lambda & & & & \downarrow \eta \\ & & & & G(F(G(I) \otimes G(A))) \\ & & & & \downarrow G(n) \\ G(A) & \xleftarrow{G(\lambda)} & G(I \otimes A) & \xleftarrow{G(\epsilon \otimes \epsilon)} & G(F(G(I)) \otimes F(G(A))) \end{array}$$

commutes. This is an exercise in patience using naturality and the triangle axioms. See the diagram below:

$$\begin{array}{ccccccc} I \otimes G(A) & \xrightarrow{\eta \otimes \text{id}} & GF(I) \otimes G(A) & \xrightarrow{G(d) \otimes \text{id}} & G(I) \otimes G(A) & & \\ \downarrow \eta & & \downarrow \eta & & \downarrow \eta & & \\ GF(I \otimes G(A)) & \xrightarrow{GF(\eta \otimes \text{id})} & GF(GF(I) \otimes G(A)) & \xrightarrow{GF(G(d) \otimes \text{id})} & GF(G(I) \otimes G(A)) & & \\ \downarrow G(n) & & \downarrow G(n) & & \downarrow G(n) & & \\ G(F(I) \otimes FG(A)) & \xrightarrow{GF(\eta \otimes \text{id})} & GFG(F(I) \otimes FG(A)) & \xrightarrow{GF(G(d) \otimes \text{id})} & G(FG(I) \otimes FG(A)) & & \\ & \searrow \text{id} & \downarrow G(\epsilon \otimes \text{id}) & & \downarrow G(\epsilon \otimes \text{id}) & & \\ & & G(F(I) \otimes FG(A)) & \xrightarrow{G(d \otimes \text{id})} & G(I \otimes FG(A)) & & \\ & & & & \downarrow G(1 \otimes \epsilon) & & \\ & & & & G(I \otimes A) & & \\ & & & & \downarrow G(\lambda) & & \\ & & & & G(A) & & \\ \lambda \nearrow & & \eta \uparrow & & \nearrow G(\epsilon) & & \\ & & G(A) & \xrightarrow{\text{id}} & G(A) & & \end{array}$$

The compatibility with the associativity is similar although slightly less absurd looking.

- (b) If G is lax symmetric monoidal, can you say anything about F ? Give a proof or counterexample.

SOLUTION: Replacing F and G with their opposite functors $F: \mathcal{D}^{\text{op}} \rightarrow \mathcal{C}^{\text{op}}$ and $G: \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}^{\text{op}}$ switches which functor is which adjoint and switches lax and oplax. Thus, using the argument above, if a right adjoint is lax symmetric monoidal, the left adjoint is oplax symmetric monoidal. In fact, the procedures of obtaining one structure from the other are inverse; given an adjunction, the data of an oplax structure on the left adjoint is bijective to the data of a lax structure on the right adjoint.

- (2) Every spectrum X is an S -module. Describe the $\pi_* S$ -action on $\pi_* X$.

SOLUTION: Since π_* is lax monoidal, there are maps

$$\pi_* S \otimes \pi_* X \rightarrow \pi_*(S \wedge X) \cong \pi_* X$$

making π_*X into a π_*S -module. Given $\alpha \in \pi_n S$ and $\xi \in \pi_m X$, we can describe $\alpha \cdot \xi$ as follows. Suppose that α is represented by the homotopy class of a map $\alpha: S^{n+k} \rightarrow S^k$ and ξ is represented by the homotopy class of a map $\chi: S^{m+\ell} \rightarrow X_\ell$. To produce the class of a map in $\pi_{n+m} X$, consider the diagram:

$$\begin{array}{ccc} S^{n+m+k+\ell} & \xrightarrow{\alpha \cdot \chi} & X_{k+\ell} \\ \downarrow \cong & & \uparrow \sigma \\ S^{n+m} \wedge S^{k+\ell} & \xrightarrow{\chi_{m,k}} S^{n+k} \wedge S^{m+\ell} \xrightarrow{\alpha \wedge f} S^k \wedge X_\ell \end{array}$$

The map $\alpha \cdot \chi$ is the composite of the maps around the outside of the rectangle, where σ is an iterated structure map of the spectrum X . The shuffle permutation $\chi_{m,k}$ is a map of degree $(-1)^{mk}$ on $S^{n+m+k+\ell}$, so the total composite is

$$\alpha \cdot \chi = [\sigma \circ (\alpha \wedge \chi) \circ (-1)^{mk}].$$

(3) Let R be a commutative ring spectrum.

- (a) Show that the forgetful functor from the category of R -module spectra to symmetric (or orthogonal) spectra has a left adjoint.

SOLUTION: A symmetric ring spectrum is a monoid object in the category of symmetric spectra, and a module over it is a module object. Writing Sp^Σ for the category of symmetric spectra, the left adjoint of the forgetful functor $\mathrm{Mod}_R \rightarrow \mathrm{Sp}^\Sigma$ is the functor $\mathrm{Sp}^\Sigma \rightarrow \mathrm{Mod}_R: X \mapsto R \wedge X$. The module structure on $R \wedge X$ comes from the multiplication $R \wedge R \rightarrow R$.

Let X be a symmetric spectrum and M an R -module. To each map $X \rightarrow M$ of spectra, we may associate a map $R \wedge X \rightarrow R \wedge M \rightarrow M$ where the second map is the action map for M . This is in fact a map of R -module spectra, as witnessed by the square

$$\begin{array}{ccccc} R \wedge R \wedge X & \longrightarrow & R \wedge R \wedge M & \longrightarrow & R \wedge M \\ \downarrow & & \downarrow & & \downarrow \\ R \wedge X & \longrightarrow & R \wedge M & \longrightarrow & M \end{array}$$

where the right square commutes because M is a module.

Further, to each map $R \wedge X \rightarrow M$ of R -module spectra, we may associate a map $X \xrightarrow{\cong} S \wedge X \rightarrow R \wedge X \rightarrow M$ using the unit of R . These two associations are natural in X and M , and in fact, they are inverses of each other because of the unitality axioms. This establishes the desired adjunction. The argument for orthogonal spectra is identical.

- (b) Let M be an R -module spectrum such that $\pi_* M$ is free as a graded $\pi_* R$ -module. Show that M is stably equivalent to a wedge sum of shifts of copies of R .

SOLUTION: Suppose the generators of $\pi_* M$ as an R -module are x_i in degree d_i . These generators produce maps $S[-d_i] \xrightarrow{x_i} M$ which, on homotopy, send the generator of $\pi_* S[-d_i]$ to x_i . By the previous part, these maps correspond to R -module homomorphisms $R[-d_i] \cong R \wedge S[-d_i] \rightarrow M$. By construction, this sends the $\pi_* R$ -module generator of $\pi_* R[-d_i]$ to x_i . We may wedge these maps together to obtain a map

$$\bigvee_i R[-d_i] \xrightarrow{\bigvee_i x_i} M$$

which is an isomorphism on homotopy groups by construction. Thus, M is stably equivalent to a wedge of shifts of R .

REFERENCES

- [Mal23] Cary Malkiewich. Spectra and stable homotopy theory. http://people.math.binghamton.edu/malkiewich/spectra_book_draft.pdf, October 2023.