

Reading: Good introductions to operads include the following: [Bel17, Bra17, Sta04, Sar17]. Pick whichever reference that you find most accessible. The survey article [Man22] is a much more comprehensive overview of operads in stable homotopy theory, but it is also much more technical.

- (1) Let X be a grouplike E_k -algebra in $\mathcal{T}op_*$ for some $k \geq 2$. Show that $\pi_0(X)$ and $\pi_1(X)$ are abelian groups.
- (2) Let $(\mathcal{C}, \otimes, I)$ and $(\mathcal{D}, \odot, J)$ be symmetric monoidal categories enriched in $\mathcal{T}op_*$, and let $\mathcal{O}$ be an operad of pointed spaces. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a lax symmetric monoidal functor.
 - (a) If X is an $\mathcal{O}$ -algebra in $\mathcal{C}$, show that $F(X)$ is an $\mathcal{O}$ -algebra in $\mathcal{D}$.
 - (b) If F is strong monoidal with right adjoint $G: \mathcal{D} \rightarrow \mathcal{C}$, show that $F \dashv G$ induces an adjunction

$$F: \text{Alg}_{\mathcal{O}}(\mathcal{C}) \rightleftarrows \text{Alg}_{\mathcal{O}}(\mathcal{D}): G$$

between the categories of $\mathcal{O}$ -algebras in $\mathcal{C}$ and $\mathcal{O}$ -algebras in $\mathcal{D}$.

- (3) Let $\widehat{\mathcal{S}p}_{\geq 0}$ be the full subcategory of $\widehat{\mathcal{S}p}$ on the connective spectra.
 - (a) Show that there is an adjunction $\pi_0: \text{ho}(\widehat{\mathcal{S}p}_{\geq 0}) \rightleftarrows \mathcal{A}b: H$.
 - (b) Let A be an abelian group. Show that the endomorphism operads $\text{End}_{\widehat{\mathcal{S}p}}(HA)$ and $\text{End}_{\mathcal{A}b}(A)$ are weakly equivalent as operads in $\mathcal{T}op_*$.
 - (c) Show that the structure of an $\mathcal{O}$ -algebra on HA is equivalent to the structure of an $\pi_0(\mathcal{O})$ -algebra on A , where $\pi_0(\mathcal{O})$ is the operad in $(\text{Set}, \times, \{*\})$ constructed from $\mathcal{O}$ by taking $\pi_0(\mathcal{O})(n) = \pi_0(\mathcal{O}(n))$.
 - (d) If A is a commutative ring, show that HA is an E_{∞} -algebra.

REFERENCES

- [Bel17] Eva Belmont. A quick introduction to operads. <https://mathweb.ucsd.edu/~ebelmont/operads-talk.pdf>, 2017.
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- [Man22] Michael A. Mandell. Operads and operadic algebras in homotopy theory. In *Stable categories and structured ring spectra*, volume 69 of *Math. Sci. Res. Inst. Publ.*, pages 183–247. Cambridge Univ. Press, Cambridge, 2022.
- [Sar17] Maru Sarazola. Loop spaces and operads. <https://sites.google.com/view/msarazola/notes-and-other-resources>, 2017.
- [Sta04] Jim Stasheff. What is ... an operad? *Notices Amer. Math. Soc.*, 51(6):630–631, 2004.