

Reading: [vK13, Sections 3 and 4] or [Bou79, Sections 1 and 2]. I also found these slides of Aras Ergus helpful [Erg19].

- (1) Let $\mathcal{S}p_Q$ be the full subcategory of $\mathcal{S}p$ on the Q -local spectra (the rational spectra).
 - (a) Show that if R is a ring spectrum, any R -module is R -local.
 - (b) Show that any Q -local spectrum is an HQ -module in the stable homotopy category.
 - (c) Show that any map of Q -local spectra is automatically a map of HQ -modules in the stable homotopy category.
 - (d) Conclude that $ho(\mathcal{S}p_Q)$ is equivalent to the category of HQ -modules in $ho(\mathcal{S}p)$.
- (2) Let $\widehat{\mathcal{S}p}$ be your favorite symmetric monoidal category of spectra (e.g. symmetric or orthogonal spectra), and let $\widehat{\mathcal{S}p}_E$ be the full subcategory of $\widehat{\mathcal{S}p}$ on the E -local spectra.
 - (a) If $f: W \rightarrow X$ and $g: Y \rightarrow Z$ are E -equivalences, show that

$$L_E(W \wedge Y) \xrightarrow{L_E(f \wedge g)} L_E(X \wedge Z)$$

is a stable equivalence.

- (b) Define $X \wedge^E Y := L_E(X \wedge Y)$. Show that $\wedge^E$ defines a symmetric monoidal structure on $ho(\widehat{\mathcal{S}p}_E)$ with unit $L_E(S)$.
 - (c) Conclude that L_E is a strong monoidal functor and the composite $ho(\widehat{\mathcal{S}p}) \xrightarrow{L_E} ho(\widehat{\mathcal{S}p}_E) \xrightarrow{\iota} ho(\widehat{\mathcal{S}p})$ is lax symmetric monoidal. Hence, $L_E(S)$ is always a commutative monoid in the stable homotopy category.
- (3) The *Bousfield class* of a spectrum E is the set of E -acyclic spectra, denoted $\langle E \rangle$. The set of Bousfield classes of spectra forms a poset with $\langle E \rangle \geq \langle D \rangle$ if being E -acyclic implies being D -acyclic.
 - (a) Show that $\langle * \rangle$ is a maximum and $\langle S \rangle$ is a minimum in this poset.
 - (b) Show that if $\langle E \rangle \geq \langle D \rangle$, then there is a natural map $L_E X \rightarrow L_D X$.
 - (c) Show that if $\langle E \rangle \geq \langle D \rangle$, then $L_D L_E X \simeq L_D X$.

REFERENCES

- [Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.
- [Erg19] Aras Ergus. The localization of spectra with respect to homology by A. K. Bousfield, eCHT Kan Seminar 2019. <https://www.aergus.net/academic/documents/assorted/bousfield-localization.pdf>, 2019.
- [vK13] Paul van Koughnett. Spectra and localization. https://people.math.harvard.edu/~hirolee/pretalbot2013/notes/2013-02-07-Paul-VanKoughnett-Bousfield_Localization.pdf, 2013.

Credit for all problems to Bert Guillou.