

Reading: [vK13, Sections 3 and 4] or [Bou79, Sections 1 and 2]. I also found these slides of Aras Ergus helpful [Erg19].

- (1) Let $\mathcal{S}p_Q$ be the full subcategory of $\mathcal{S}p$ on the Q -local spectra (the rational spectra).
 - (a) Show that if R is a ring spectrum, any R -module is R -local.
 - (b) Show that any Q -local spectrum is an HQ -module in the stable homotopy category.
 - (c) Show that any map of Q -local spectra is automatically a map of HQ -modules in the stable homotopy category.
 - (d) Conclude that $ho(\mathcal{S}p_Q)$ is equivalent to the category of HQ -modules in $ho(\mathcal{S}p)$.

SOLUTION:

- (a) Suppose M is an R -module spectrum. To show that M is R -local, it suffices to show that for any R -acyclic module A we have

$$[A, M] = 0$$

To show this, let $f : A \rightarrow M$ and consider the following diagram:

$$\begin{array}{ccccc} A & \xrightarrow{f} & M & & \\ i \wedge id_A \downarrow & & i \wedge id_M \downarrow & \searrow & \\ R \wedge A & \xrightarrow{id_R \wedge f} & R \wedge M & \xrightarrow{\mu} & M \end{array}$$

where i is the unit map $i : S \rightarrow R$. This diagram commutes. Along the top, we have the map f , so the same map is the composition along the bottom. However, since A is R -acyclic, we know that $R \wedge A \simeq *$, so f factors through $*$, and thus must be trivial. This is it.

- (b) Let L_{HQ} be the localization with respect to HQ . In fact, we know that this is a smashing localization, so $L_{HQ}X = HQ \wedge X$ for a given spectrum X .

If a spectrum Z is Q -local, in particular this implies that the localization of Z is equivalent to Z , and there is a map $\eta_Z : Z \rightarrow L_{HQ}Z = HQ \wedge Z$ realizing this. In the homotopy category, weak equivalences are inverted, so we may take our module action map $\alpha : HQ \wedge Z \rightarrow Z$ to be the inverse of the above weak equivalence.

It remains to check that the proper diagrams commute. However, this is probably obvious.

- (c) Starting with a map $f : Z_1 \rightarrow Z_2$ of Q -local spectra, consider the image of f under the natural transformation $\eta : id \Rightarrow L_{HQ}$

$$\begin{array}{ccc} Z_1 & \xrightarrow{f} & Z_2 \\ \eta_{Z_1} \downarrow & & \downarrow \eta_{Z_2} \\ HQ \wedge Z_1 & \xrightarrow{1 \wedge f} & HQ \wedge Z_2 \end{array}$$

This diagram commutes (up to homotopy, by naturality of η). Since the action is defined through taking a homotopy inverse of η , f is a HQ -module homomorphism.

- (d) This is an immediate consequence of parts (b) and (c).

- (2) Let $\widehat{\mathcal{S}p}$ be your favorite symmetric monoidal category of spectra (e.g. symmetric or orthogonal spectra), and let $\widehat{\mathcal{S}p}_E$ be the full subcategory of $\widehat{\mathcal{S}p}$ on the E -local spectra.

- (a) If $f: W \rightarrow X$ and $g: Y \rightarrow Z$ are E -equivalences, show that

$$L_E(W \wedge Y) \xrightarrow{L_E(f \wedge g)} L_E(X \wedge Z)$$

is a stable equivalence.

- (b) Define $X \wedge^E Y := L_E(X \wedge Y)$. Show that $\wedge^E$ defines a symmetric monoidal structure on $\text{ho}(\widehat{\mathcal{S}p}_E)$ with unit $L_E(S)$.
- (c) Conclude that L_E is a strong monoidal functor and the composite $\text{ho}(\widehat{\mathcal{S}p}) \xrightarrow{L_E} \text{ho}(\widehat{\mathcal{S}p}_E) \xrightarrow{\iota} \text{ho}(\widehat{\mathcal{S}p})$ is lax symmetric monoidal. Hence, $L_E(S)$ is always a commutative monoid in the stable homotopy category.

SOLUTION:

- (a) By the E -Whitehead Theorem, it suffices to show that $L_E(f \wedge g)$ is an E -equivalence. This will follow if we can show that $f \wedge g$ is an E -equivalence, since L_E preserves E -homology.

Recall that a map is an E -equivalence iff the fiber is E -acyclic, and observe that $f \wedge g = (1_Y \wedge g) \circ (f \wedge 1_Y)$. Then to show that $f \wedge g$ is an E -equivalence, it suffices to show that $f \wedge 1_Y$ and $1_Y \wedge g$ are, i.e., that their fibers are E -acyclic.

Since smashing with a fixed spectrum is a left adjoint, it preserves fibers, and thus $\text{fib}(f \wedge 1_Y) \simeq \text{fib}(f) \wedge Y$. Now since $\text{fib}(f)$ is E -acyclic, we have $E \wedge \text{fib}(f) \simeq *$, and it follows that $\text{fib}(f \wedge 1_Y)$ is E -acyclic as well. Therefore $f \wedge 1_Y$ is an E -equivalence, and the proof that $1_Y \wedge g$ is an E -equivalence is essentially identical.

- (b) Recall that $X \rightarrow L_E X$ is an E -equivalence, for any spectrum X . If X is E -local, then this is a stable equivalence.

For the unit, apply part (a) with $f: S \rightarrow L_E S$ and $g = \text{id}_X$. This gives a stable equivalence

$$L_E(X) = L_E(S \wedge X) \simeq L_E(L_E S \wedge X) = L_E S \wedge^E X$$

Composing with $X \simeq L_E(X)$ yields a stable equivalence $X \simeq L_E S \wedge^E X$. Similarly, $X \simeq X \wedge^E L_E S$.

For the associativity, apply part (a) with $f: X \wedge Y \rightarrow L_E(X \wedge Y)$ and $g = \text{id}_Z$. This gives a stable equivalence

$$L_E(X \wedge Y \wedge Z) \rightarrow L_E(L_E(X \wedge Y) \wedge Z) = (X \wedge^E Y) \wedge^E Z$$

Similarly, $X \wedge^E (Y \wedge^E Z)$ is stably equivalent to $L_E(X \wedge Y \wedge Z)$, and therefore stably equivalent to $(X \wedge^E Y) \wedge^E Z$.

I won't check the coherence axioms here.

- (c) The maps that define the strong monoidal structure on L_E are $\text{id}: L_E S \rightarrow L_E S$ and

$$L_E X \wedge^E L_E Y \rightarrow L_E(X \wedge Y)$$

coming from the inverse (in $\text{ho}(\widehat{\mathcal{S}p}_E)$) of the stable equivalence deduced from part (a) using $f: X \rightarrow L_E(X)$ and $g: Y \rightarrow L_E(Y)$.

It is a tedious but straightforward exercise to check the right diagrams commute.

- (3) The *Bousfield class* of a spectrum E is the set of E -acyclic spectra, denoted $\langle E \rangle$. The set of Bousfield classes of spectra forms a poset with $\langle E \rangle \geq \langle D \rangle$ if being E -acyclic implies being D -acyclic.

- (a) Show that $\langle * \rangle$ is a maximum and $\langle S \rangle$ is a minimum in this poset.
- (b) Show that if $\langle E \rangle \geq \langle D \rangle$, then there is a natural map $L_E X \rightarrow L_D X$.

(c) Show that if $\langle E \rangle \geq \langle D \rangle$, then $L_D L_E X \simeq L_D X$.

SOLUTION:

- (a) Every spectrum X is $*$ -acyclic, because $* \wedge X \simeq *$. Therefore, being E -acyclic implies being $*$ -acyclic for any E , so $\langle * \rangle$ is a minimum in this poset.
On the other hand, a spectrum X is S -acyclic if and only if $S \wedge X \simeq *$. So only $*$ is S -acyclic. So X being S -acyclic implies that X is E -acyclic for any E . So $\langle S \rangle$ is a maximum.
- (b) If $\langle E \rangle \geq \langle D \rangle$, then any E -acyclic spectrum is D -acyclic. Claim that $L_D X$ is E -local, so it admits a map from the initial E -localization $L_E X$ of X . To see that $L_D X$ is E -local, we must show that for any E -acyclic A , $[A, L_D X] = 0$. But if A is E -acyclic, then A is also D -acyclic, so $[A, L_D X] = 0$ because $L_D X$ is D -local. Hence, $L_D X$ is E -local. Therefore, there is a map $L_E X \rightarrow L_D X$ from the initial E -localization of X .
- (c) The map $X \rightarrow L_E X$ is an E -equivalence, which means that the fiber F of this map is E -acyclic. Since $\langle E \rangle \geq \langle D \rangle$, this means the fiber is D -acyclic, which is equivalent to the map $X \rightarrow L_E X$ being a D -equivalence. Then applying L_D to this map yields $L_D X \rightarrow L_D L_E X$, which is a D -equivalence between D -local spectra. By the D -Whitehead theorem, this is a stable equivalence.

REFERENCES

- [Bou79] A. K. Bousfield. The localization of spectra with respect to homology. *Topology*, 18(4):257–281, 1979.
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- [vK13] Paul van Koughnett. Spectra and localization. https://people.math.harvard.edu/~hirolee/pretalbot2013/notes/2013-02-07-Paul-VanKoughnett-Bousfield_Localization.pdf, 2013.

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