

CHROMATIC HOMOTOPY TALK OUTLINE

This outline is only a suggestion. There are many different directions that you could go with this talk. Use it as an excuse to learn about something you're curious about, and if you would like to talk about something other than what's in the outline below, feel free to do so!

(1) MU and friends

- (a) The spectrum MU is the version of the spectrum MO using the infinite unitary group U instead of the infinite orthogonal group O. As a cohomology theory, it represents complex coborsism.
- (b) More precisely, let BU(n) be the classifying space of the unitary group U(n). This is the space of n-planes in $\mathbb{C}^\infty$. Let $\gamma_n \rightarrow BU(n)$ be the tautological bundle of n-planes in $\mathbb{C}^\infty$. The spectrum MU has

$$MU_{2n} = \text{Th}(\gamma^n), \quad MU_{2n+1} = \Sigma MU_{2n}.$$

The map $BU(n) \rightarrow BU(n+1)$ gives double suspension between the Thom spaces because of the difference between complex dimension and real dimension.

- (c) It is a theorem due to Milnor and independently Novikov that $\pi_* MU \cong \mathbb{Z}[x_1, x_2, \dots]$ with $|x_i| = 2i$.
- (d) We can localize at p (i.e. take $L_{S(p)} = (-) \wedge S_{(p)}$, see [Mal23, Example 2.5.31]). It turns out (see [Mal23, Example 2.5.33]) that this localization has the effect of localizing $\pi_* MU$ at the ideal (p):

$$\pi_* MU_{(p)} = \mathbb{Z}_{(p)}[x_1, x_2, \dots]$$

- (e) Quillen proved that there is an idempotent map $e: MU_{(p)} \rightarrow MU_{(p)}$ and this defines a summand of $MU_{(p)}$ by an old homework problem. We define the *Brown Peterson spectrum* BP as the summand of $MU_{(p)}$ that splits off

$$MU_{(p)} \simeq BP \vee \text{other stuff}$$

We omit the p from the notation for the Brown Peterson spectrum, but it's implicitly there in the background. Also it turns out that the other stuff is more copies of BP, shifted appropriately. [Mal23, Example 2.5.39]

- (f) The homotopy of BP is

$$\pi_* BP = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$$

where $|v_i| = 2(p^i - 1)$.

- (g) Remember that we can form Postnikov towers of spectra to kill off certain homotopy groups. By taking Postnikov towers, we form the *truncated Brown Peterson spectra* $BP\langle n \rangle$, with

$$\pi_* BP\langle n \rangle \cong \mathbb{Z}_{(p)}[v_1, \dots, v_n]$$

To be precise, $BP\langle n \rangle = P_{2(p^n-1)}BP$ as a Postnikov stage (see [Mal23, Definition 2.6.30]). Beware that the $\langle n \rangle$ in $BP\langle n \rangle$ is different than the $\langle n \rangle$ in the Whitehead tower from [Mal23, bottom of page 118].

In particular, notice that $BP\langle 0 \rangle = H\mathbb{Z}_{(p)}$.

- (h) For any ring spectrum R and any $f \in \pi_n R$, we can form a new spectrum $R[f^{-1}]$ by “inverting f.” In detail, $f: \Sigma^n S \rightarrow R$ gives, by smashing with R and composing with multiplication, a map $f: \Sigma^n R \rightarrow R \wedge R \rightarrow R$. Then we can iterate suspensions of this map and take a homotopy colimit to get

$$R[f^{-1}] = \text{hocolim}(R \xrightarrow{f} \Sigma^{-n} R \xrightarrow{f} \Sigma^{-2n} R \rightarrow \dots)$$

and $\pi_*(R[f^{-1}]) = (\pi_*R)[f^{-1}]$ (invert f in the graded ring π_*R). This will be an exercise on Thursday.

Applying this procedure to $BP\langle n \rangle$, we can form a spectrum $E(n)$, the *Johnson-Wilson spectrum* by inverting $v_n \in \pi_{2(p^n-1)}BP\langle n \rangle$.

$$\pi_*E(n) = \mathbb{Z}_{(p)}[v_1, \dots, v_n][v_n^{-1}] \text{ for } n \geq 1$$

By convention, $v_0 = p$, so $E(0) = HQ$. ($\mathbb{Z}_{(p)}$ inverts all integers that are not multiples of p , and then inverting p gives $\mathbb{Q}$)

- (i) Another algebraic construction: Let R be a ring spectrum. A sequence of elements $f_1, f_2, \dots$ in π_*R is called a *regular sequence* if multiplication by f_n is injective on the quotient $(\pi_*R)/(f_1, \dots, f_{n-1})$ for all n .

Given an element $f \in \pi_*R$, we have a map $f: \Sigma^n S \rightarrow R$. Smashing with R and composing with multiplication gives a map $f: \Sigma^n R \rightarrow R \wedge R \rightarrow R$. We write R/f for the cofiber of $f: \Sigma^n R \rightarrow R$.

Given a regular sequence $f_1, f_2, \dots \in \pi_*R$, write $R/(f_1, f_2) = (R/f_1)/f_2$, etc. Then it is a fact (which will be an exercise on Thursday)

$$\pi_*(R/(f_1, f_2, \dots)) \cong (\pi_*R)/(f_1, f_2, \dots).$$

Applying this procedure to the regular sequence $(p, v_1, \dots, v_{n-1})$ in $\pi_*E(n)$, we can form a spectrum $K(n)$

$$K(n) = E(n)/(p, v_1, \dots, v_{n-1})$$

such that

$$\pi_*K(n) = \mathbb{Z}/p[v_n^{\pm 1}].$$

These are called the *Morava K-theory spectra*. Note that $K(0) = E(0) = HQ$.

(2) A reminder about localization

- (a) Recall: let E be a spectrum. If A is any other spectrum such that $E_*A = \pi_*(E \wedge A) = 0$, then A is called *E-acyclic*. If X satisfies $[A, X] = 0$ for any E -acyclic spectrum A , we say that X is *E-local*.
- (b) We saw two weeks ago that there is a functor $L_E: \mathcal{S}p \rightarrow \mathcal{S}p_E$ from the category of spectra to the category of E -local spectra, and L_E is the initial E -local spectrum. It comes with a natural transformation $\text{id}_{\mathcal{S}p} \rightarrow L_E$.
- (c) For example, HQ -localization is rationalization (tensor homotopy groups with HQ). S/p -localization is p -completion (tensor homotopy groups of X with the p -adic integers, if $\pi_k X$ is finitely generated for all k).

(3) The Chromatic Tower (see [Gui24, Section 9] for most of the material in this section)

- (a) Fact: every $E(n-1)$ -local spectrum is also $E(n)$ -local, so there is a natural transformation from $L_{E(n)} \rightarrow L_{E(n-1)}$, because $L_{E(n)}$ is the initial $E(n)$ -localization.
- (b) For any spectrum X , we can write the $E(n)$ -localizations as a tower

$$X_Q = L_{E(0)}X \leftarrow L_{E(1)}X \leftarrow L_{E(2)}X \leftarrow \cdots \leftarrow L_{E(n)}X \leftarrow \cdots \leftarrow X.$$

This is the *chromatic tower* of X . Each term has a map from X coming from the natural transformations $\text{id} \rightarrow L_{E(n)}$.

- (c) **The Big Theorem** State the Chromatic Convergence Theorem, due to Ravenel: If X is a finite p -local spectrum, then $X \simeq \operatorname{holim}_n L_{E(n)}X$. (Remember there are p s in the $E(n)$, but they are hidden by the notation. So the p -local assumption isn't that weird.)

A consequence of this theorem is that we can learn a lot about a spectrum X by learning about the $E(n)$ -localizations of X .

- (d) To understand the $E(n)$ -localizations of S , we use a homotopy pullback/pushout square called the *chromatic fracture square*. The following is a homotopy pullback/pushout:

$$\begin{array}{ccc} L_{E(n)}X & \longrightarrow & L_{E(n-1)}X \\ \downarrow & & \downarrow \\ L_{K(n)}X & \longrightarrow & L_{E(n-1)}L_{K(n)}X \end{array}$$

- (e) State the Smash Product Theorem of Hopkins–Ravenel that $L_{E(n)}$ is a smashing localization:

$$L_{E(n)}X \simeq L_{E(n)}S \wedge X.$$

So if we want to understand the chromatic tower of X , we can get pretty far by first understanding the chromatic tower of S .

- (f) Applying the Smash Product Theorem to the chromatic fracture square, we get

$$\begin{array}{ccc} L_{E(n)}S \wedge X & \longrightarrow & L_{E(n-1)}S \wedge X \\ \downarrow & & \downarrow \\ L_{K(n)}X & \longrightarrow & L_{E(n-1)}S \wedge L_{K(n)}X \end{array}$$

So to understand the chromatic tower for X in going from the $E(n-1)$ -localization to the $E(n)$ -localization, the key thing is to understand $L_{K(n)}X$.

- (g) In particular, to understand the chromatic tower for S , we want to understand the chromatic tower for $L_{K(n)}S$. This is not easy! We only really know $L_{K(1)}S$ really well, but people have done lots of (very hard) work too understand $L_{K(2)}S$. Recently, Barthel-Schlank-Stapleton-Weinstein proved that

$$\pi_*((L_{K(n)}S)_Q) \cong \Lambda_Q(\zeta_1, \dots, \zeta_n)$$

is an exterior algebra over Q with generators $\zeta_1, \dots, \zeta_n$ with $|\zeta_i| = 1 - 2i$. Their proof is hardcore number theory.

- (h) If you want, and you have time, state [Gui24, Theorem 9.3].

REFERENCES

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